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Calculus of Variations

The Einstein-scalar field Lichnerowicz equations on graphs

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Abstract
In this article, we consider the Einstein-scalar field Lichnerowicz equation

$$-\Delta u + hu = Bu^{p-1} + Au^{-p-1}$$

on any connected finite graph $G = (V, E)$, where A, B, h are given functions on V with $A \geq 0$, $A \not\equiv 0$ on V, and $p > 2$ is a constant. By using the classical variational method, topological degree theory and heat-flow method, we provide a systematical study on this equation by providing the existence results for each case: positive, negative and null Yamabe-scalar field conformal invariant, namely $h > 0$, $h < 0$ and $h = 0$ respectively.

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1 Introduction

In general relativity, $(\mathbf{V}, \mathbf{g})$ stands for a spacetime with manifold $\mathbf{V} = M \times \mathbb{R}$, which is a Cauchy development of the geometric initial data $(M, \overline{g}, \overline{K})$, where $(M, \overline{g})$ is an n-dimensional Riemannian manifold and $\overline{K}$ is a $(0, 2)$-tensor. The spacetime metric $\mathbf{g}$ satisfies the Einstein equation

$$\mathrm{Ric}_{\mathbf{g}} - \frac{1}{2}\mathrm{Scal}_{\mathbf{g}}\mathbf{g} = T,$$

where T is a symmetric $(0, 2)$-tensor, $\mathrm{Ric}_{\mathbf{g}}$ and $\mathrm{Scal}_{\mathbf{g}}$ are the Ricci tensor and the scalar curvature of the spacetime metric $\mathbf{g}$ respectively. In addition, $(M, \overline{g}, \overline{K})$ should be embedded isometrically into $(\mathbf{V}, \mathbf{g})$ as a slice with the second fundamental form $\overline{K}$. Thus, the initial data $(\overline{g}, \overline{K})$ must satisfy the following constraint equations (see [2, 4, 20])

$$\mathcal{H}(\overline{g}, \overline{K}) \equiv \mathrm{Scal}_{\overline{g}} - |\overline{K}|^2_{\overline{g}} + \left(\mathrm{trace}_{\overline{g}}\overline{K}\right)^2 - 2\rho = 0, \tag{1.1}$$

and

$$\mathcal{M}(\overline{g}, \overline{K}) \equiv \nabla_{\overline{g}}\overline{K} - \nabla_{\overline{g}}\left(\mathrm{trace}_{\overline{g}}\overline{K}\right) - J = 0, \tag{1.2}$$

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where $\mathrm{Scal}_{\overline{g}}$ is the scalar curvature of $\overline{g}$, ρ is the energy density and J is the momentum density of the nongravitational field, denoted by $\rho = T(\mathbf{n}, \mathbf{n})$ and $J = -T(\mathbf{n}, \cdot)$, where $\mathbf{n}$ is the unit timelike normal to the slice $M \times \{0\}$, see [2, 5]. Then the unknowns in (1.1) and (1.2) are the metric $\overline{g}$ and tensor $\overline{K}$.

In the constant mean curvature (CMC) case, by applying a conformal change, one turns the constraints (1.1) and (1.2) to the so-called Hamiltonian and momentum constraints. Precisely, for a given metric g on M, we look for some smooth scalar function φ with

$$\overline{g}_{ij} = \varphi^{\frac{4}{n-2}} g_{ij}, \quad \overline{K}_{ij} = \frac{\tau}{n}\varphi^{\frac{4}{n-2}} g_{ij} + \varphi^{-2}(\sigma + \mathcal{L}_g W)_{ij},$$

satisfying the constraints (1.1) and (1.2). Here $\mathcal{L}_g$ denotes the conformal killing operator acting on W defined by $\mathcal{L}_g W_{ij} := W_{i,j} + W_{j,i} - \frac{2}{n}\mathrm{div}_g W g_{ij}$, τ is the mean curvature of M computed with respect to $\overline{g}$, and σ is a transverse and traceless tensor. So the Hamiltonian constraint (1.1) becomes a semi-linear elliptic equation of (φ, W),

$$\Delta_g \varphi + \mathcal{R}_\psi \varphi = \mathcal{B}_{\tau,\psi,U}\varphi^{2^*-1} + \mathcal{A}_{\pi,\sigma}(W)\varphi^{-2^*-1}, \tag{1.3}$$

where

$$\mathcal{R}_\psi = k_n\big(\mathrm{Scal}_g - |\nabla\psi|_g^2\big), \quad \mathcal{A}_{\pi,\sigma}(W) = k_n\big(|\sigma + \mathcal{L}_g W|_g^2 + \pi^2\big),$$

and

$$\mathcal{B}_{\tau,\psi,U} = -k_n\left(\frac{n-1}{n}\tau^2 - 2U(\psi)\right),$$

where $\Delta_g = -\mathrm{div}_g(\nabla_g)$ is the Laplace-Beltrami operator, $2^* = \frac{2n}{n-2}$ is the critical Sobolev exponent, $k_n = \frac{n-2}{4(n-1)}$, Scal_g is the scalar curvature relative to g, ψ is a real scalar field on the spacetime $(\mathbf{V}, \mathbf{g})$, and U is a potential function of ψ. In addition, the momentum constraint (1.2) turns to be

$$\mathrm{div}_g\left(\mathcal{L}_g W\right) = \frac{n-1}{n}\varphi^{2^*}\nabla\tau - \pi\nabla\psi. \tag{1.4}$$

In the CMC setting, $\nabla\tau \equiv 0$ and (1.4) implies that $W \equiv 0$, then the system (1.3) and (1.4) is semi-decoupled associated to (φ, W). It remains to solve Eq. (1.3). We usually write (1.3) as

$$\Delta_g u + hu = Bu^{2^*-1} + Au^{-2^*-1} \text{ on } M, \tag{1.5}$$

with $u = \varphi$, $h = \mathcal{R}_\psi$, $A = \mathcal{A}_{\pi,\sigma}(W)$ and $B = \mathcal{B}_{\tau,\psi,U}$. Actually, Eq. (1.5) is the Einstein-scalar field Lichnerowicz equation on Riemannian manifold with $h, A, B \in C^\infty(M)$ satisfying $A \geq 0$.

For the case of $h < 0$ being a constant, Ngô-Xu [19] obtained some existence results for positive solution to (1.5) when appropriately adjusting coefficients h, A, B, and discussed the uniqueness property under some additional conditions. For the case of $h \leq 0$, we refer to [5, 12, 15] for more interesting work. For the case of $h = 0$, Ngô-Xu [20] established some existence and uniqueness results to (1.5) under different assumptions on A and B. Ma et al. [17] introduced the heat-flow method to address the null case as well. For the case of $-\Delta + h$ being coercive (e.g., $\max_M h > 0$ or $h > 0$ in M), see Hebey-Pacard-Pollack [9] and Ma-Wei [18] for some variational arguments. Interested readers are referred to [3, 6, 7, 21] for more results on Eq. (1.5).

Recently, there has been growing interest among mathematicians in exploring the mathematical and physical equations on graphs. These equations include the mean field equation

([10, 11]) and the Kazdan–Warner equation ([8, 13, 16, 23]). The focus of this attention has been on issues such as the existence, uniqueness, stability, and topological degree results associated with these equations. For instance, Huang et al. [10] recently established the existence of solutions for the mean field equation on a connected finite graph, Sun and Wang [23] provided novel proofs for some previously known results regarding the existence and multiplicity of solutions for the Kazdan–Warner equation on a connected finite graph by employing Brouwer degree computations. By computing the topological degree and using the relation between the degree and the critical group of a related functional for the Chern-Simons Higgs models on the finite graph, Li et al. [14] derived some interesting result for the multiple solutions of the system. Inspired by the work of [9, 14, 17–19, 23], we consider the Einstein-scalar field Lichnerowicz Eq. (1.5) on any connected finite graph $G = (V, E)$, that is,

$$-\Delta u + hu = Bu^{p-1} + Au^{-p-1} \text{ on } V, \tag{1.6}$$

where $p > 2$, A, B, h are given functions on V satisfying that $A \geq 0$ and $A \not\equiv 0$ on V. For simplicity, we always call equations of the form (1.6) the *EL* equation.

This article is organized as follows. In Sect. 2, we review some settings of a graph and give our main results. In Sect. 3, we present some lemmas that will be used in subsequent sections, including Sobolev embedding, Maximum Principles and a blow-up analysis result. In Sect. 4, for the positive case, we establish some existence and multiplicity results by variational method and calculate the topological degree for Eq. (1.6) under certain mild assumptions. In Sect. 5, our focus shifts to the negative case, specifically when the constant h is negative. Here, we analyze the asymptotic functional and derive a positive solution for Eq. (1.6), with strictly negative energy. Lastly, in Sect. 6, we employ a heat-flow method to obtain a positive solution for Eq. (1.6) in the null case and compute its associated topological degree.

We denote by $u^{\pm}$ the positive/negative part of $u \in V^{\mathbb{R}}$ that stands for $u^{\pm} = \max\{\pm u, 0\}$.

2 Notations and main results

We explain some settings and represent our main results in this section. Let $G = (V, E)$ be a connected finite graph with $m := \text{Card}(V) < +\infty$. We use positive numbers ω_{xy} to represent the weights of edges, for any $x, y \in V$, $\omega_{xy} > 0$ if and only if $xy \in E$. In other words, $\omega_{xy} = 0$ means $xy \notin E$. In addition, the weight is symmetric, $\omega_{xy} = \omega_{yx}$ for any $xy \in E$. Let μ be a positive finite measure on V. Denote by $|V| = \sum_{x \in V} \mu(x)$ the volume of V. Let $V^{\mathbb{R}}$ be the vector space of all real functions on V. For any $u \in V^{\mathbb{R}}$, we define the Laplacian operator by

$$\Delta u(x) = \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}\big(u(y) - u(x)\big), \quad \forall\, x \in V,$$

where $y \sim x$ means $xy \in E$. As usual, we define $\mu(x) = \sum_{y \sim x} \omega_{xy}$ for any $x \in V$, and then Δ is the normalized Laplacian operator. If $\mu(x) \equiv 1$ for any $x \in V$, then Δ is the combinatorial graph Laplacian operator (see [16]). For any $u, v \in V^{\mathbb{R}}$, the gradient form of u and v is given by

$$\Gamma(u, v)(x) = \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}\big(u(y) - u(x)\big)\big(v(y) - v(x)\big), \quad \forall\, x \in V.$$

For simplicity, we write $\Gamma(u)(x) = \Gamma(u,u)(x)$ and $\nabla u \cdot \nabla v = \Gamma(u,v)$. Define the length of gradient of u as $|\nabla u|(x) = \sqrt{\Gamma(u)}(x)$ for any $x \in V$. For any $u \in V^{\mathbb{R}}$, the integral of u over V is defined by

$$\int_V u(x)\mathrm{d}\mu = \sum_{x\in V} \mu(x)u(x).$$

Define the Lebesgue spaces $L^q(V) = \big\{u \in V^{\mathbb{R}} \mid \sum_{x\in V} \mu(x)|u(x)|^q < \infty\big\}$ for any $q \in (0,+\infty)$, and $L^\infty(V) = \big\{u \in V^{\mathbb{R}} \mid \max_{x\in V} |u(x)| < \infty\big\}$. If $h \in V^{\mathbb{R}}$ is assumed to be positive, then we shall consider Eq. (1.6) in the following Sobolev space

$$H_h^1(V) := \Big\{u \in V^{\mathbb{R}} \mid \int_V \big(|\nabla u|^2 + hu^2\big)\mathrm{d}\mu < \infty\Big\},$$

equipped with the norm

$$\|u\|_{H_h^1(V)} = \left(\int_V \big(|\nabla u|^2 + hu^2\big)\mathrm{d}\mu\right)^{\frac{1}{2}}, \quad \text{for } u \in H_h^1(V).$$

In particular, if $h(x) \equiv 1$ on V, $H_h^1(V)$ is written as $W^{1,2}(V)$ as usual.

Definition 2.1 We call $u \in V^{\mathbb{R}}$ a weak positive solution to (1.6) if $u \in W^{1,2}(V)$ with $u > 0$ on V, and the integral identity

$$\int_V \big(\Gamma(u,v) + huv\big)\mathrm{d}\mu = \int_V v\big(Bu^{p-1} + Au^{-p-1}\big)\mathrm{d}\mu$$

holds for any $v \in W^{1,2}(V)$. We say $u \in V^{\mathbb{R}}$ is a point-wise positive solution to (1.6) if $u \in L^\infty(V)$ with $u > 0$ on V, and

$$-\Delta u(x) + h(x)u(x) = B(x)u(x)^{p-1} + A(x)u(x)^{-p-1}, \quad \forall\, x \in V.$$

Remark 1 (a) One can easily check that for any $u, v \in V^{\mathbb{R}}$, the integration by parts formula holds

$$\int_V (-\Delta u)v\mathrm{d}\mu = \int_V \Gamma(u,v)\mathrm{d}\mu = \int_V \Gamma(v,u)\mathrm{d}\mu = \int_V u(-\Delta v)\mathrm{d}\mu.$$

(b) Supposing that $G = (V, E)$ is a connected finite graph, then the definitions of positive solution to (1.6) between weak sense and point-wise sense are equivalent. In fact, we just apply Lemma 3.1 and choose the test function δ_{x_0} for any $x_0 \in V$ as follows

$$\delta_{x_0}(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0. \end{cases} \tag{2.1}$$

In order to present our results, we introduce the topological degree for Eq. (1.6). For any $u \in L^\infty(V)$ with $u > 0$ on V, we denote the associated energy functional by

$$\mathcal{J}(u) = \frac{1}{2}\int_V \big(|\nabla u|^2 + hu^2\big)\mathrm{d}\mu - \frac{1}{p}\int_V B(x)u^p\mathrm{d}\mu + \frac{1}{p}\int_V A(x)u^{-p}\mathrm{d}\mu.$$

We consider the map

$$\mathcal{A}_{h,A,B} : L_+^\infty(V) \to L^\infty(V), \quad u \mapsto -\Delta u + hu - Bu^{p-1} - Au^{-p-1},$$

where $L_+^\infty(V) = \{u \in L^\infty(V) \mid u > 0 \text{ on } V\}$ that can be viewed as an open subset in $\mathbb{R}^m$. Denote by $B_R = \{u \in L^\infty(V) \mid 0 < u < R\} \subseteq L_+^\infty(V)$ that can be treated as an open subset as well. For the positive case, we conclude by Lemma 3.3 and Proposition 4.1 that the Brouwer degree $\deg(\mathcal{A}_{h,A,B}, B_R, 0)$ is well defined for $R > 0$ large. While for the null case, we refer the readers to Lemma 6.6. Then by the homotopic invariance, $\deg(\mathcal{A}_{h,A,B}, B_R, 0)$ is independent of R. We define the topological degree by

$$\mathbf{d}_{h,A,B} = \lim_{R\to+\infty} \deg(\mathcal{A}_{h,A,B}, B_R, 0).$$

Then

$$\deg(\mathcal{A}_{h,A,B}, B_R, 0) = \sum_{u\in B_R, \mathcal{A}_{h,A,B}(u)=0} \operatorname{sgn} \det(D\mathcal{A}_{h,A,B}(u)),$$

whenever $\partial B_R \cap \mathcal{A}^{-1}_{h,A,B}(\{0\}) = \emptyset$. We refer the readers to [1, 23] for the corresponding definition.

Now we state our main results in this article. First, for the positive case, as in [18, Theorem 1], applying the monotone method, we have

Theorem 2.2 *Suppose that $G = (V, E)$ is a connected finite graph, and $h(x) > 0$, $A(x) > 0$, $B(x) \geq 0$ on V. If $\overline{u} \in H^1_h(V)$ is a positive super-solution to (1.6), then for sufficiently small $\delta > 0$, there exists a positive solution u to Eq. (1.6) satisfying that $\delta \leq u \leq \overline{u}$ on V.*

Denote by

$$p^\diamond = \left(\frac{p+2}{p-2}\right)^{\frac{p-2}{2p}} + \left(\frac{p+2}{p-2}\right)^{-\frac{p+2}{2p}}.$$

Under some mild assumptions, our next result involves the topological degree for the positive case.

Theorem 2.3 *Let $G = (V, E)$ be a connected finite graph, $h(x) > 0$, $A(x) > 0$ and $B(x) > 0$ on V. Suppose that*

$$A(x) \leq A_0, \quad B(x) \leq B_0, \quad h(x) \geq h_0, \quad \forall\, x \in V, \tag{2.2}$$

for some positive constants A_0, B_0 and h_0.

(a) If there exists a positive solution S to the following equation

$$h_0 s - B_0 s^{p-1} - A_0 s^{-p-1} = 0, \tag{2.3}$$

then for sufficiently small $\delta > 0$, there is a positive solution u to (1.6) such that $\delta \leq u(x) \leq S$ on V.

(b) If $A(x) \equiv A_0$, $B(x) \equiv B_0$, $h(x) \equiv h_0$ on V, and $h_0 = A_0^{\frac{p-2}{2p}} B_0^{\frac{p+2}{2p}} p^\diamond$, then Eq. (1.6) admits only the constant solution

$$u_1(x) \equiv \left(\frac{A_0(p+2)}{B_0(p-2)}\right)^{\frac{1}{2p}}, \quad \text{for any } x \in V.$$

(c) Let $G = (V, E)$ be a complete graph, that is $xy \in E$ for any $x, y \in V$. Suppose that $h(x)$, $A(x)$ and $B(x)$ are not all constants on V, and $h_0 \geq A_0^{\frac{p-2}{2p}} B_0^{\frac{p+2}{2p}} p^\diamond$, then Eq. (1.6) admits at least two positive solutions, and the topological degree $\mathbf{d}_{h,A,B} = 0$.

As in [9, Theorem 3.1], applying the variational method, we obtain a mountain pass solution.

Theorem 2.4 *Suppose that $G=(V,E)$ is a connected finite graph, and $h(x)>0$, $A(x)>0$, $B(x)>0$ on V. There exists a positive function $\zeta\in V^{\mathbb{R}}$ with $\|\zeta\|_{H_h^1(V)}=1$ and a constant $C=C(p)>0$ depending only on p such that if*

$$\int_V A(x)\zeta^{-p}\mathrm{d}\mu \le C\Big(S_h^p \max_{x\in V} B(x)\Big)^{-\frac{p+2}{p-2}}, \tag{2.4}$$

then the EL Eq. (1.6) admits a positive solution, where S_h stands for the Sobolev embedding constant corresponding to $H_h^1(V)\hookrightarrow L^p(V)$.

For the negative case, we apply some similar variational analysis of [19, Theorem 1.1] and obtain a positive solution with negative energy.

Theorem 2.5 *Let $G=(V,E)$ be a connected finite graph. Assume that $A, B, h\in V^{\mathbb{R}}$ and*

(a) $h(x)\equiv h<0$ on V, where h is a constant and $|h|<\lambda_B$. Here $\lambda_B>0$ is a positive constant defined by (5.4) and (5.5).
(b) $A(x)\ge 0$, $A(x)\not\equiv 0$ on V, and

$$\int_V A(x)\mathrm{d}\mu \le \left(\frac{p+2}{4}\frac{|h|}{\int_V B^-(x)\mathrm{d}\mu}\right)^{\frac{p+2}{p-2}} \frac{|h|(p-2)|V|^{\frac{2p}{p-2}}}{4}. \tag{2.5}$$

(c) $\max_{x\in V} B(x)>0$, and $\int_V B(x)\mathrm{d}\mu<0$.

Then there exists some constant $\Upsilon_2>0$ (see (5.30) in Lemma 5.10) such that if

$$\max_{x\in V} B(x) < \Upsilon_2\int_V B^-(x)\mathrm{d}\mu, \tag{2.6}$$

then the EL Eq. (1.6) admits at least one positive solution with negative energy.

Finally, we give our main results for the null case. We consider the heat flow (see [17])

$$\begin{cases} u_t-\Delta u = g(x,u), & \text{in } V\times(0,+\infty),\\ u(x,0)=u_0(x), & \text{on } V,\end{cases} \tag{2.7}$$

where $u_0(x)$ is an arbitrary positive function and $g(x,u)=B(x)u^{p-1}+A(x)u^{-p-1}$ with $p>2$.

Theorem 2.6 *Let $G=(V,E)$ be a connected finite graph. Suppose that $h(x)\equiv 0$, $A(x)>0$, $B(x)<0$ and the initial data $u_0(x)>0$ on V. Then there exists a unique positive solution $u(x,t)\in C^\infty\big([0,\infty);V^{\mathbb{R}}\big)$ to (2.7). In addition, $u(x,t)\to u_\infty(x)$ in $L^\infty(V)$ as $t\to+\infty$ suitably, that is, there is a subsequence $\{t_k\}$ with $t_k\to+\infty$ as $k\to+\infty$ such that*

$$u(x,t_k)\to u_\infty(x) \text{ uniformly on } V,\quad \text{as } k\to+\infty, \tag{2.8}$$

where $u_\infty(x)$ is a positive solution to the EL Eq. (1.6), that is,

$$-\Delta u_\infty = B(x)u_\infty^{p-1}+A(x)u_\infty^{-p-1} \text{ on } V. \tag{2.9}$$

The last conclusion gives the topological degree for null case.

Theorem 2.7 *Let $G=(V,E)$ be a complete finite graph and $h(x)\equiv 0$ on V. Suppose that $A(x)>0$ and $B(x)<0$ are not all constants on V. Then (1.6) admits at least one positive solution and the topological degree $\mathbf{d}_{0,A,B}=1$.*

Let us close this section by mentioning the new ingredients and the distinctions between our current work and previous studies conducted on manifolds. The compact nature of the embedding in the case of finite graphs simplifies most of the problems, making them more direct and manageable. In particular, when dealing with the elliptic problem, we automatically get a strictly lower bound for the solution, which alleviates the challenge of defining the associated energy functional that involves the term $A(x)u^{-p-1}$. This a-priori estimate enables us to calculate the associated topological degree, ensuring the existence of multiple solutions. Furthermore, it is well-established that in the classical context, $\int |\nabla|u||^2 = \int |\nabla u|^2$. Though it is true that $\int |\nabla|u||^2 \leq \int |\nabla u|^2$ for the graph case, we can not show the corresponding energy functional is differentiable. Consequently, we are constrained to work with non-negative functions, necessitating a departure from the approach used in [19] to obtain a second solution. In the case of the parabolic problem, defining the corresponding sub-solution and super-solution requires us to solve the associated heat equation on the graph, which inherently involves a system of ODEs. To derive the existence result, we have to pay some attention on the computations involving the term u_t. Besides, we rely on the equation to justify the uniqueness and regularity. While the underlying principles remain consistent with the classical setting in spirit, the techniques employed turn to be slightly different due to the distinct nature of the problems on graph.

3 Preliminaries

In this section, we present some preliminary results that will be used in subsequent parts. Here, we denote by $h_{\min} = \min_{x\in V} h(x)$ if $h(x) > 0$ on V, and $\mu_{\min} = \min_{x\in V} \mu(x)$ in this part.

Lemma 3.1 (Sobolev embedding) *Suppose that $G = (V, E)$ is a connected finite graph and $h(x) > 0$ on V. Then*

(a) the Sobolev embedding $H_h^1(V) \hookrightarrow L^q(V)$ is continuous for $q \in [1, +\infty]$, and there exists some positive constant S_h depending on h, G and q such that for any $u \in H_h^1(V)$, it holds that

$$\|u\|_{L^q(V)} \leq S_h \|u\|_{H_h^1(V)}. \tag{3.1}$$

(b) the Sobolev embedding $H_h^1(V) \hookrightarrow L^q(V)$ is compact for $q \in [1, +\infty]$. Furthermore, the boundedness and precompactness are equivalent in $H_h^1(V)$ and $L^q(V)$ for $q \in [1, +\infty]$, respectively.

Proof (a) Suppose that $u \in H_h^1(V)$. For any $x_0 \in V$, direct computation shows that

$$\begin{aligned}\|u\|^2_{H_h^1(V)} &= \int_V \left(\Gamma(u) + hu^2\right) \mathrm{d}\mu \geq h_{\min} \int_V u^2 \mathrm{d}\mu = h_{\min} \sum_{x\in V} \mu(x) u^2(x) \\ &\geq h_{\min}\mu_{\min} u^2(x_0),\end{aligned}$$

which implies that

$$|u(x_0)| \leq (h_{\min}\mu_{\min})^{-\frac{1}{2}} \|u\|_{H_h^1(V)}.$$

Therefore, $H_h^1(V) \hookrightarrow L^\infty(V)$ is continuous and

$$\|u\|_{L^\infty(V)} \leq S_h \|u\|_{H_h^1(V)}, \quad \text{where } S_h = (h_{\min}\mu_{\min})^{-\frac{1}{2}}. \tag{3.2}$$

Now for any $q \in [1, +\infty)$, we have

$$\|u\|_{L^q(V)}^q = \sum_{x\in V} \mu(x)|u(x)|^q \leq \sum_{x\in V} \mu(x) \left(h_{\min}\mu_{\min}\right)^{-\frac{q}{2}} \|u\|_{H_h^1(V)}^q$$
$$= |V| \left(h_{\min}\mu_{\min}\right)^{-\frac{q}{2}} \|u\|_{H_h^1(V)}^q.$$

Therefore, $H_h^1(V) \hookrightarrow L^q(V)$ is continuous for any $q \in [1, +\infty)$ and

$$\|u\|_{L^q(V)} \leq S_h \|u\|_{H_h^1(V)}, \quad \text{where} \quad S_h = |V|^{\frac{1}{q}} \left(h_{\min}\mu_{\min}\right)^{-\frac{1}{2}}. \tag{3.3}$$

Combining (3.2) and (3.3), we finish the proof of (a).

(b) Let $\{u_j\}$ be a bounded sequence in $H_h^1(V)$. After passing to a subsequence if necessary, we may assume that $u_j \rightharpoonup u$ in $H_h^1(V)$. Since h is positive on V, $\{u_j\}$ is bounded in $L^2(V)$. Then up to a subsequence (still denoted by $\{u_j\}$), $u_j \rightharpoonup u$ in $L^2(V)$. Thus, for any $v \in L^2(V)$, we have

$$\lim_{j\to+\infty} \int_V (u_j - u)v\mathrm{d}\mu = \lim_{j\to+\infty} \sum_{x\in V} \mu(x)\big(u_j(x) - u(x)\big)v(x) = 0. \tag{3.4}$$

For any $x_0 \in V$, we substitute the test function $v = \delta_{x_0}$, (defined as in (2.1)), into (3.4) and then get

$$\lim_{j\to+\infty} \mu(x_0)\big(u_j(x_0) - u(x_0)\big) = 0,$$

which yields that

$$\lim_{j\to+\infty} u_j(x_0) = u(x_0).$$

As a consequence, we have $u_j \to u$ in $L^q(V)$ for $q \in [1, +\infty]$. Therefore, the embedding $H_h^1(V) \hookrightarrow L^q(V)$ is compact for $q \in [1, +\infty]$. Furthermore, by the finiteness of V, we obtain

$$\lim_{j\to+\infty} \|\nabla u_j - \nabla u\|_{L^2(V)}^2 = \lim_{j\to+\infty} \int_V \Gamma(u_j - u)\mathrm{d}\mu$$
$$= \frac{1}{2} \lim_{j\to+\infty} \sum_{x\in V} \sum_{y\sim x} \omega_{xy} \Big(\big(u_j(y) - u(y)\big) - \big(u_j(x) - u(x)\big)\Big)^2 = 0,$$

and

$$\lim_{j\to+\infty} \int_V h(x)\big(u_j(x) - u(x)\big)^2\mathrm{d}\mu = \lim_{j\to+\infty} \sum_{x\in V} \mu(x)h(x)\big(u_j(x) - u(x)\big)^2 = 0.$$

Thus, we derive the precompactness from the boundedness in $H_h^1(V)$. On the other hand, if $u_j \to u$ in $H_h^1(V)$, one can directly check that $\{u_j\}$ is bounded in $H_h^1(V)$. Hence, the boundedness and precompactness are equivalent in $H_h^1(V)$. For $L^q(V)$, $q \in [1, +\infty]$, the argument is similar and we omit the details. This finishes the proof of conclusion (b). □

Next, we establish the Maximum Principle for the elliptic equation on finite graph.

Lemma 3.2 (Maximum Principle) *Suppose that $G = (V, E)$ is a connected finite graph and $h(x) > 0$ on V. A function $u \in V^{\mathbb{R}}$ is said to be a super- (sub-) solution of $-\Delta u + hu = 0$ if it holds that*

$$-\Delta u(x) + h(x)u(x) \geq (\leq)\, 0, \quad \forall\, x \in V. \tag{3.5}$$

(a) If u is a super-solution of $-\Delta u + hu = 0$, then $u(x) \geq 0$ on V. Furthermore, it must hold

$$\text{either } \ u \equiv 0 \ \text{ or } \ u > 0 \ \text{ on } V.$$

(b) If u is a sub-solution of $-\Delta u + hu = 0$, then $u(x) \leq 0$ on V. Furthermore, it must hold

$$\text{either } \ u \equiv 0 \ \text{ or } \ u < 0 \ \text{ on } V.$$

Proof (a) We prove the conclusion by contradiction. Suppose that this is not true and there exists $x_0 \in V$ such that $\min_{x\in V} u(x) = u(x_0) < 0$. Then we obtain $-\Delta u(x_0) \leq 0$ and $h(x_0)u(x_0) < 0$. From this, we deduce a contradiction to (3.5). Thus, $u(x) \geq 0$ on V. Furthermore, if there exists $x_1 \in V$ such that $\min_{x\in V} u(x) = u(x_1) = 0$, then we get

$$0 \geq -\Delta u(x_1) = -\Delta u(x_1) + h(x_1)u(x_1) \geq 0,$$

which implies that

$$0 = -\Delta u(x_1) = -\frac{1}{\mu(x_1)} \sum_{y\sim x_1} \omega_{x_1 y}\big(u(y) - u(x_1)\big) \leq 0.$$

Therefore, $u(y) = u(x_1) = 0$ for any $y \sim x_1$. Since G is connected and finite, by induction, we conclude that $u(x) \equiv 0$ on V. This finishes the proof of (a).

(b) Applying (a) to $-u$, we get the desired conclusion.

□

Remark 2 In general, for $h(x) > 0$ on V, we call function $u \in V^{\mathbb{R}}$ a super-solution of $-\Delta u + hu = 0$ on V if for any $\psi \in H_h^1(V)$ with $\psi \geq 0$, it holds that

$$\int_V \big(\Gamma(u, \psi) + hu\psi\big)\mathrm{d}\mu \geq 0. \tag{3.6}$$

For any $x_0 \in V$, substituting the test function $\psi = \delta_{x_0}$ (defined as in (2.1)) into (3.6), we have

$$-\Delta u(x_0) + h(x_0)u(x_0) \geq 0.$$

As a consequence, the definitions of super-solution are equivalent between weak sense and point-wise sense. The same conclusion holds for sub-solutions.

Lemma 3.3 *Suppose that $A, B, h \in V^{\mathbb{R}}$, $A(x) \geq 0$ and $A(x) \not\equiv 0$ on V. Let $\{u_n\}$ be a sequence of positive solutions to (1.6), namely,*

$$-\Delta u_n(x) + h_n(x)u_n(x) = B_n(x)u_n(x)^{p-1} + A_n(x)u_n(x)^{-p-1}, \quad \forall\, x \in V,$$

where $\{A_n\}$, $\{B_n\}$ and $\{h_n\}$ satisfy that

$$\lim_{n\to+\infty} A_n(x) = A(x), \quad \lim_{n\to+\infty} B_n(x) = B(x), \quad \lim_{n\to+\infty} h_n(x) = h(x), \quad \forall\, x \in V.$$

Assume that $\{u_n\}$ is uniformly bounded from below by a positive constant. Then, up to a subsequence, one of the following alternatives holds

(a) *either* $\{u_n\}$ *is uniformly bounded (that is, bounded in* $L^\infty(V)$*), or*
(b) *there exists* $x_0 \in V$ *such that* $u_n(x_0)$ *converges to* $+\infty$ *and* $B(x_0) = 0$*. Moreover,* $\{u_n\}$ *is uniformly bounded from above in* $\Omega := \{x \in V \mid B(x) > 0\}$*.*

Proof If $\{u_n\}$ is uniformly bounded from above, then $\{u_n\}$ is uniformly bounded, and hence (a) holds true. While if $\limsup_{n\to+\infty} \max_{x\in V} u_n(x) \to +\infty$, we may assume that there is a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and some $x_0 \in V$ such that

$$u_n(x_0) = \max_{x\in V} u_n(x) \to +\infty, \quad \text{as } n \to +\infty. \tag{3.7}$$

On the other hand, since $\{u_n\}$ is uniformly bounded from below by a positive constant, for any $\overline{x} \in V$ we have

$$\begin{aligned} B_n(\overline{x})u_n(\overline{x})^{p-1} - h_n(\overline{x})u_n(\overline{x}) &= -\Delta u_n(\overline{x}) - A_n(\overline{x})u_n(\overline{x})^{-p-1} \\ &\le \frac{1}{\mu(\overline{x})} \sum_{y\sim\overline{x}} \omega_{\overline{x}y}(u_n(\overline{x}) - u_n(y)) + 0 \\ &\le u_n(\overline{x}) + C, \end{aligned}$$

which implies that

$$B_n(\overline{x}) - h_n(\overline{x})u_n(\overline{x})^{2-p} \le (u_n(\overline{x}) + C)\, u_n(\overline{x})^{1-p}.$$

Letting $n \to +\infty$, we deduce that $B(\overline{x}) \le 0$ whenever $\limsup_{n\to+\infty} u_n(\overline{x}) \to +\infty$, where we have used that $p > 2$. Hence $\{u_n\}$ is uniformly bounded in $\Omega = \{x \in V \mid B(x) > 0\}$. Thus by (3.7), we have $B(x_0) \le 0$.

Next we prove that $B(x_0) \ge 0$. Using the fact that x_0 is a maximum point of $u_n(x)$ on V we have

$$\begin{aligned} B_n(x_0)u_n(x_0)^{p-1} &= -\Delta u_n(x_0) + h_n(x_0)u_n(x_0) - A_n(x_0)u_n(x_0)^{-p-1} \\ &\ge h_n(x_0)u_n(x_0) - A_n(x_0)u_n(x_0)^{-p-1}, \end{aligned}$$

which implies that

$$B_n(x_0) \ge h_n(x_0)u_n(x_0)^{2-p} - A_n(x_0)u_n(x_0)^{-2p}.$$

Letting $n \to +\infty$, we deduce that $B(x_0) \ge 0$. Thus, $B(x_0) = 0$. This finishes the proof of Lemma 3.3. □

4 Variational analysis and topological degree for the positive case

In this section, we consider the positive solution of the *EL* Eq. (1.6), namely,

$$-\Delta u + hu = Bu^{p-1} + Au^{-p-1} \text{ on } V, \tag{4.1}$$

where $A, B, h \in V^{\mathbb{R}}$, $h(x) > 0$, $A(x) \ge 0$ and $A(x) \not\equiv 0$ on V unless otherwise specified.

Proposition 4.1 *Suppose that* $A(x) > 0$ *and* $B(x) > 0$ *on* V*. If*

$$\max_{x\in V} \frac{h(x)}{A(x)^{\frac{p-2}{2p}} B(x)^{\frac{p+2}{2p}}} < p^\diamond,$$

then Eq. (4.1) *does not possess any positive solution.*

Proof Suppose that u is a positive solution to (4.1). Letting $x_0 \in V$ be a minimum point of u on V, then we have $-\Delta u(x_0) \leq 0$ and

$$h(x_0) \geq B(x_0)u(x_0)^{p-2} + A(x_0)u(x_0)^{-p-2} > A(x_0)u(x_0)^{-p-2},$$

which implies that

$$u(x) \geq u(x_0) > \left(\frac{A(x_0)}{h(x_0)}\right)^{\frac{1}{p+2}} \geq \left(\min_{x\in V} \frac{A(x)}{h(x)}\right)^{\frac{1}{p+2}}, \quad \forall\, x \in V. \tag{4.2}$$

It follows that any positive solution to (4.1) is uniformly bounded from below by a positive constant. Furthermore, denote by

$$f(t) = B(x_0)t + A(x_0)t^{-\frac{p+2}{p-2}}, \ t > 0.$$

One can easily obtain that $\min_{t>0} f(t) = f(t_0)$, where

$$t_0 = \left(\frac{A(x_0)(p+2)}{B(x_0)(p-2)}\right)^{\frac{p-2}{2p}}, \quad f(t_0) = A(x_0)^{\frac{p-2}{2p}} B(x_0)^{\frac{p+2}{2p}} p^{\diamond}.$$

Hence, there must exist some $x \in V$ such that

$$h(x) \geq A(x)^{\frac{p-2}{2p}} B(x)^{\frac{p+2}{2p}} p^{\diamond}.$$

Contradiction arises. This finishes the proof of Proposition 4.1. □

Example 1 Let $G = (V, E)$ be a connected finite graph. Then the following *EL* equation

$$-\Delta u + u = u^{p-1} + u^{-p-1} \text{ on } V, \tag{4.3}$$

does not possess any positive solution. In fact, by Proposition 4.1, the nonexistence is trivial. We can also obtain this result by the a priori estimate (4.2). Indeed, supposing that u is a positive solution to (4.3), we choose some minimum point x_0 of u on V. Then by (4.2), we find that $u(x) \geq u(x_0) \geq 1$ for any $x \in V$. However,

$$u(x_0) \geq -\Delta u(x_0) + u(x_0) = u(x_0)^{p-1} + u(x_0)^{-p-1} > u(x_0)^{p-1} \geq u(x_0), \text{ since } p > 2.$$

Then contradiction arises and the desired conclusion holds.

Applying similar arguments of [9, Theorems 2.1 and 2.2], we get the following two nonexistence results, Propositions 4.2 and 4.3.

Proposition 4.2 *Suppose that* $B(x) > 0$ *on* V *and*

$$\left(\int_V h^{\frac{p-1}{p-2}} B^{-\frac{1}{p-2}} \mathrm{d}\mu\right)^{\frac{p-2}{p-1}} < p^{\diamond} \left(\int_V A^{\frac{p-1}{2p}} B^{\frac{p+1}{2p}} \mathrm{d}\mu\right)^{\frac{p-2}{p-1}}. \tag{4.4}$$

Then Eq. (4.1) *does not possess any positive solution.*

Proof Suppose that (4.1) admits a positive solution u and (4.4) holds. Integrating Eq. (4.1) on both sides, we obtain

$$\int_V hu\mathrm{d}\mu = \int_V Bu^{p-1}\mathrm{d}\mu + \int_V Au^{-p-1}\mathrm{d}\mu. \tag{4.5}$$

By Hölder's inequality we get

$$\int_V hu\mathrm{d}\mu \leq \left(\int_V h^{\frac{p-1}{p-2}} B^{-\frac{1}{p-2}} \mathrm{d}\mu\right)^{\frac{p-2}{p-1}} \left(\int_V Bu^{p-1}\mathrm{d}\mu\right)^{\frac{1}{p-1}},$$

and

$$\int_V A^{\frac{p-1}{2p}} B^{\frac{p+1}{2p}} \mathrm{d}\mu \leq \left(\int_V Bu^{p-1}\mathrm{d}\mu\right)^{\frac{p+1}{2p}} \left(\int_V Au^{-p-1}\mathrm{d}\mu\right)^{\frac{p-1}{2p}}.$$

Combining the above two inequalities with (4.5), we have

$$\begin{aligned}\left(\int_V Bu^{p-1}\mathrm{d}\mu\right)^{\frac{p-2}{p-1}} + \left(\int_V A^{\frac{p-1}{2p}} B^{\frac{p+1}{2p}} \mathrm{d}\mu\right)^{\frac{2p}{p-1}} \left(\int_V Bu^{p-1}\mathrm{d}\mu\right)^{-\frac{p+2}{p-1}} \\ \leq \left(\int_V h^{\frac{p-1}{p-2}} B^{-\frac{1}{p-2}} \mathrm{d}\mu\right)^{\frac{p-2}{p-1}}.\end{aligned}$$

Set $\widetilde{f}(t) = t + Kt^{-\frac{p+2}{p-2}}$ for $t > 0$, where

$$t = \left(\int_V Bu^{p-1}\mathrm{d}\mu\right)^{\frac{p-2}{p-1}} \quad \text{and} \quad K = \left(\int_V A^{\frac{p-1}{2p}} B^{\frac{p+1}{2p}} \mathrm{d}\mu\right)^{\frac{2p}{p-1}}.$$

One can easily check that

$$\min_{t>0} \widetilde{f}(t) = \widetilde{f}(t_1) = p^{\diamond} K^{\frac{p-2}{2p}} = p^{\diamond} \left(\int_V A^{\frac{p-1}{2p}} B^{\frac{p+1}{2p}} \mathrm{d}\mu\right)^{\frac{p-2}{p-1}},$$

where

$$t_1 = \left(K\frac{p+2}{p-2}\right)^{\frac{p-2}{2p}} = \left(\frac{p+2}{p-2}\right)^{\frac{p-2}{2p}} \left(\int_V A^{\frac{p-1}{2p}} B^{\frac{p+1}{2p}} \mathrm{d}\mu\right)^{\frac{p-2}{p-1}}.$$

Therefore, we deduce that

$$p^{\diamond} \left(\int_V A^{\frac{p-1}{2p}} B^{\frac{p+1}{2p}} \mathrm{d}\mu\right)^{\frac{p-2}{p-1}} \leq \left(\int_V h^{\frac{p-1}{p-2}} B^{-\frac{1}{p-2}} \mathrm{d}\mu\right)^{\frac{p-2}{p-1}},$$

which contradicts to (4.4). This finishes the proof of Proposition 4.2. □

Remark 3 Proposition 4.2 is a general version of Proposition 4.1. By Proposition 4.1, if we assume that

$$h(x) < A(x)^{\frac{p-2}{2p}} B(x)^{\frac{p+2}{2p}} p^{\diamond}, \quad \forall\, x \in V, \tag{4.6}$$

then Eq. (4.1) does not admit any positive solution. We can rewrite (4.6) as

$$h(x)^{\frac{p-1}{p-2}} B(x)^{-\frac{1}{p-2}} < \left(p^{\diamond}\right)^{\frac{p-1}{p-2}} A(x)^{\frac{p-1}{2p}} B(x)^{\frac{p+1}{2p}}. \tag{4.7}$$

Integrating (4.7) on both sides we obtain (4.4), since V is finite. Particularly, Example 1 also provides an example for Proposition 4.2.

Proposition 4.3 *Suppose that $G = (V, E)$ is a connected finite graph and*

$$S_h^p \kappa^p \left(\max_{x \in V} B^-(x) + S_h^{-p} \kappa^{2-p} \right)^{\frac{1}{2}} < \int_V A^{\frac{1}{2}}(x) \mathrm{d}\mu \tag{4.8}$$

for some constant $\kappa > 0$, where S_h is the Sobolev embedding constant for $H_h^1(V) \hookrightarrow L^p(V)$ (Lemma 3.1-(a) with $q = p$). Then Eq. (4.1) does not possess positive solutions satisfying $\|u\|_{H_h^1(V)} \leq \kappa$.

Proof Suppose that u is a positive solution to (4.1) satisfying $\|u\|_{H_h^1(V)} \leq \kappa$. Multiplying (4.1) by u and integrating on both sides, we get

$$\int_V B u^p \mathrm{d}\mu + \int_V A u^{-p} \mathrm{d}\mu = \|u\|_{H_h^1(V)}^2 \leq \kappa^2. \tag{4.9}$$

Applying the Sobolev's inequality (3.1) with $q = p$, we obtain

$$\int_V B u^p \mathrm{d}\mu \geq -S_h^p \kappa^p \max_{x \in V} B^-(x),$$

which together with (4.9) implies that

$$\int_V A u^{-p} \mathrm{d}\mu \leq \kappa^2 + S_h^p \kappa^p \max_{x \in V} B^-(x). \tag{4.10}$$

On the other hand, by Hölder's inequality, it holds that

$$\int_V A^{\frac{1}{2}}(x) \mathrm{d}\mu \leq \left(\int_V A u^{-p} \mathrm{d}\mu \right)^{\frac{1}{2}} \left(\int_V u^p \mathrm{d}\mu \right)^{\frac{1}{2}} \leq \left(\int_V A u^{-p} \mathrm{d}\mu \right)^{\frac{1}{2}} S_h^{\frac{p}{2}} \kappa^{\frac{p}{2}}. \tag{4.11}$$

Hence we deduce from (4.10) to (4.11) that

$$\int_V A^{\frac{1}{2}}(x) \mathrm{d}\mu \leq S_h^p \kappa^p \left(\max_{x \in V} B^-(x) + S_h^{-p} \kappa^{2-p} \right)^{\frac{1}{2}},$$

which contradicts to (4.8). This finishes the proof of Proposition 4.3. □

4.1 Monotone method solution

In this subsection, we shall find positive solutions to Eq. (4.1) by monotone method.

Proof of Theorem 2.2 For sufficiently small constant $\delta > 0$, $u \equiv \delta$ is a sub-solution to (4.1). In fact, one can easily check that

$$h(x)\delta \leq B(x)\delta^{p-1} + A(x)\delta^{-p-1} \text{ on } V,$$

for $\delta > 0$ small enough. In the remainder of this proof, δ is fixed as above, and let $\delta < \min_{x \in V} \overline{u}(x)$ for any $x \in V$. We define the energy functional $\mathcal{I}$ corresponding to Eq. (4.1) as

$$\mathcal{I}(u) = \frac{1}{2} \int_V \left(|\nabla u|^2 + h u^2 \right) \mathrm{d}\mu - \frac{1}{p} \int_V B u^p \mathrm{d}\mu + \frac{1}{p} \int_V A u^{-p} \mathrm{d}\mu, \tag{4.12}$$

for $u \in \mathcal{N}$, where

$$\mathcal{N} = \left\{ u \in H_h^1(V) \mid \delta \leq u \leq \overline{u} \text{ on } V \right\}. \tag{4.13}$$

Notice that for any $u \in \mathcal{N}$, we have

$$\mathcal{I}(u) \geq \frac{1}{2}h_{\min}\delta^2|V| - \frac{1}{p}\max_{x\in V} B(x)\int_V \overline{u}^p \mathrm{d}\mu + \frac{1}{p}\int_V A\overline{u}^{-p}\mathrm{d}\mu,$$

which implies that $\mathcal{I}$ is bounded from below in $\mathcal{N}$. We consider the minimizing problem

$$\alpha := \inf_{u\in\mathcal{N}} \mathcal{I}(u) > -\infty. \tag{4.14}$$

Suppose that $\{u_n\} \subseteq \mathcal{N}$ is a minimizing sequence of α, that is,

$$\frac{1}{2}\int_V \left(|\nabla u_n|^2 + hu_n^2\right)\mathrm{d}\mu - \frac{1}{p}\int_V Bu_n^p\mathrm{d}\mu + \frac{1}{p}\int_V Au_n^{-p}\mathrm{d}\mu = \alpha + o_n(1). \tag{4.15}$$

Notice that

$$\left|-\frac{1}{p}\int_V Bu_n^p\mathrm{d}\mu + \frac{1}{p}\int_V Au_n^{-p}\mathrm{d}\mu\right| \leq \frac{1}{p}\max_{x\in V} B(x)\int_V \overline{u}^p\mathrm{d}\mu + \frac{1}{p\delta^p}\int_V A(x)\mathrm{d}\mu.$$

Then, combining the above inequality with (4.15), we find that $\{u_n\}$ is bounded in $H_h^1(V)$. Since $\mathcal{N}$ is closed and convex, $\mathcal{N}$ is weakly closed. By Lemma 3.1-(b), up to a subsequence (still denoted by $\{u_n\}$), we may assume that there exists some $u \in \mathcal{N}$ such that $u_n \to u$ in $L^\infty(V)$ as $n \to +\infty$. Thus by (4.15), one can easily check that $\mathcal{I}(u) = \lim_{n\to+\infty} \mathcal{I}(u_n) = \alpha$, which implies that α is achieved by $u \in \mathcal{N}$.

It remains to prove that the minimizer u satisfies Eq. (4.1). To this end, we apply the same arguments with [22, Theorem 2.4]. For any $v \in H_h^1(V)$ and any $\varepsilon > 0$, we choose a test function v_ε, defined by

$$v_\varepsilon = \min\left\{\overline{u}, \max\{\delta, u + \varepsilon v\}\right\} = u + \varepsilon v + v_{1\varepsilon} - v_{2\varepsilon},$$

where

$$v_{1\varepsilon} := \max\{0, \delta - (u + \varepsilon v)\} \text{ and } v_{2\varepsilon} := \max\{0, u + \varepsilon v - \overline{u}\}.$$

Then $v_{1\varepsilon}, v_{2\varepsilon} \in H_h^1(V)$ and are nonnegative. For any $x \in V$,

$$v_\varepsilon(x) = \begin{cases} \delta, & \text{if } u(x) + \varepsilon v(x) \leq \delta, \\ u(x) + \varepsilon v(x), & \text{if } \delta < u(x) + \varepsilon v(x) \leq \overline{u}(x), \\ \overline{u}(x), & \text{if } \overline{u}(x) < u(x) + \varepsilon v(x), \end{cases}$$

which implies that $v_\varepsilon \in \mathcal{N}$. Since u is a minimizer of $\mathcal{I}$ in $\mathcal{N}$, we have $\langle \mathcal{I}'(u), v_\varepsilon - u\rangle \geq 0$, that is,

$$\langle \mathcal{I}'(u), v\rangle \geq \frac{1}{\varepsilon}\langle \mathcal{I}'(u), v_{2\varepsilon} - v_{1\varepsilon}\rangle. \tag{4.16}$$

Since $\overline{u}$ is a super-solution of (4.1), we have

$$-\Delta\overline{u}(x) + h(x)\overline{u}(x) \geq B(x)\overline{u}(x)^{p-1} + A(x)\overline{u}(x)^{-p-1}, \quad \forall\, x \in V,$$

and then

$$\langle \mathcal{I}'(\overline{u}), v_{2\varepsilon}\rangle = \int_V \left(\Gamma(\overline{u}, v_{2\varepsilon}) + h\overline{u}v_{2\varepsilon}\right)\mathrm{d}\mu - \int_V \left(B\overline{u}^{p-1} + A\overline{u}^{-p-1}\right)v_{2\varepsilon}\mathrm{d}\mu \geq 0. \tag{4.17}$$

Thus by (4.17) we have

$$\begin{aligned}
\langle \mathcal{I}'(u), v_{2\varepsilon}\rangle &= \langle \mathcal{I}'(\overline{u}), v_{2\varepsilon}\rangle + \langle \mathcal{I}'(u) - \mathcal{I}'(\overline{u}), v_{2\varepsilon}\rangle \geq \langle \mathcal{I}'(u) - \mathcal{I}'(\overline{u}), v_{2\varepsilon}\rangle \\
&= \int_V \big(\Gamma(u-\overline{u}, v_{2\varepsilon}) + h(u-\overline{u})v_{2\varepsilon}\big)\mathrm{d}\mu - \int_V \big(B(u^{p-1} - \overline{u}^{p-1}) \\
&\quad + A(u^{-p-1} - \overline{u}^{-p-1})\big)v_{2\varepsilon}\mathrm{d}\mu \\
&\geq \int_{\widetilde{V}} \big(\Gamma(u-\overline{u}, v_{2\varepsilon}) + h(u-\overline{u})v_{2\varepsilon}\big)\mathrm{d}\mu - \int_{\widetilde{V}} \big(B(u^{p-1} - \overline{u}^{p-1}) \\
&\quad + A(u^{-p-1} - \overline{u}^{-p-1})\big)v_{2\varepsilon}\mathrm{d}\mu \\
&= \int_{\widetilde{V}} \big(\Gamma(u-\overline{u}) + h(u-\overline{u})^2\big)\mathrm{d}\mu + \varepsilon \int_{\widetilde{V}} \big(\Gamma(u-\overline{u}, v) + h(u-\overline{u})v\big)\mathrm{d}\mu \\
&\quad - \int_{\widetilde{V}} \big(B(u^{p-1} - \overline{u}^{p-1}) + A(u^{-p-1} - \overline{u}^{-p-1})\big)(u + \varepsilon v - \overline{u})\mathrm{d}\mu \\
&\geq \varepsilon \int_{\widetilde{V}} \big(\Gamma(u-\overline{u}, v) + h(u-\overline{u})v\big)\mathrm{d}\mu \\
&\quad - \int_{\widetilde{V}} \big|B(u^{p-1} - \overline{u}^{p-1}) + A(u^{-p-1} - \overline{u}^{-p-1})\big|(u + \varepsilon v - \overline{u})\mathrm{d}\mu \\
&\geq \varepsilon \int_{\widetilde{V}} \big(\Gamma(u-\overline{u}, v) + h(u-\overline{u})v\big)\mathrm{d}\mu - \varepsilon \int_{\widetilde{V}} \big|B(u^{p-1} - \overline{u}^{p-1}) \\
&\quad + A(u^{-p-1} - \overline{u}^{-p-1})\big|v\mathrm{d}\mu,
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{V} &= \big\{x \in V \mid \exists\, y \in V_1 \text{ such that } xy \in E\big\} \text{ and} \\
V_1 &= \big\{x \in V \mid u(x) < \overline{u}(x) \leq u(x) + \varepsilon v(x)\big\}.
\end{aligned}$$

Since V is finite, $\widetilde{V} = \emptyset$ for $\varepsilon > 0$ sufficiently small. We conclude that

$$\langle \mathcal{I}'(u), v_{2\varepsilon}\rangle \geq 0. \tag{4.18}$$

Similarly, since δ is a sub-solution of (4.1), by the same arguments as (4.17) and (4.18), we conclude that

$$\langle \mathcal{I}'(u), v_{1\varepsilon}\rangle \leq 0. \tag{4.19}$$

Substituting (4.18) and (4.19) into (4.16), we have $\langle \mathcal{I}'(u), v\rangle \geq 0$ for any $v \in H_h^1(V)$. Taking the sign of v as minus and we get $\langle \mathcal{I}'(u), v\rangle \leq 0$. Hence it holds $\langle \mathcal{I}'(u), v\rangle = 0$. Finally, we choose the test function $v = \delta_{x_0}$ for any $x_0 \in V$, defined as in (2.1), and see that u is indeed a point-wise positive solution to (4.1). This finishes the proof of Theorem 2.2. □

Theorem 4.4 *Suppose that $G = (V, E)$ is a connected finite graph, $h(x) > 0$, $A(x) > 0$ and $B(x) \leq 0$ on V. Then Eq.* (4.1) *has at least one positive solution.*

Proof One can easily find that $\underline{u} \equiv \varepsilon$ is a sub-solution to (4.1) for $\varepsilon > 0$ sufficiently small, and $\overline{u} \equiv M$ is a super-solution to (4.1) for $M > 0$ sufficiently large. Hence by the same arguments of Theorem 2.2, we can obtain a positive solution to (4.1) by the sub- and super-solution method. □

4.2 Topological degree

We give some results on the uniqueness and multiplicity of solutions via topological degree method.

Proof of Theorem 2.3 (a) Since S is a positive solution to (2.3), we get

$$h(x)S - B(x)S^{p-1} - A(x)S^{-p-1} \geq h_0 S - B_0 S^{p-1} - A_0 S^{-p-1} = 0 = \Delta S, \quad \forall\, x \in V,$$

which implies that $\overline{u} := S$ is a positive super-solution to (4.1). Then by Theorem 2.2, we obtain the existence result.

(b) One can check easily that u_1 satisfies (1.6). We assume that $u(x)$ is another positive solution. Let $x_0 \in V$ be such that $u(x_0) = \min_{x\in V} u(x)$. Then $-\Delta u(x_0) \leq 0$ and

$$h_0 \geq B_0 u(x_0)^{p-2} + A_0 u(x_0)^{-p-2} \geq h_0,$$

which implies that $-\Delta u(x_0) = 0$. Here we have used the same computations with Proposition 4.1. Thus, $u(x)$ is equal to a positive constant, and then $u(x) \equiv u_1$ on V.

(c) Since any positive constant less than h_0 can be regarded as a positive lower bound of $h(x)$, without loss of generality, we assume that $h_0 = A_0^{\frac{p-2}{2p}} B_0^{\frac{p+2}{2p}} p^{\diamond}$. For any $t \in [0, 1]$, let u_t be a positive solution of

$$-\Delta u_t + \big((1-t)h + th_0\big)u_t = \big((1-t)B + tB_0\big)u_t^{p-1} + \big((1-t)A + tA_0\big)u_t^{-p-1} \text{ on } V. \tag{4.20}$$

Next we claim that $\{u_t\}$ is uniformly bounded on V. Suppose this is not true. Then there exists a sequence $\{t_n\} \subseteq [0, 1]$ such that $\limsup_{n\to+\infty} \max_{x\in V} u_n(x) \to +\infty$, where $u_n = u_{t_n}$ is the positive solution to

$$-\Delta u_n(x) + \widehat{h}_n(x)u_n(x) = \widehat{B}_n(x)u_n(x)^{p-1} + \widehat{A}_n(x)u_n(x)^{-p-1}, \quad \forall\, x \in V,$$

and $\{\widehat{A}_n\}$, $\{\widehat{B}_n\}$ and $\{\widehat{h}_n\}$ satisfy that

$$\widehat{A}_n = (1-t_n)A + t_n A_0, \quad \widehat{B}_n = (1-t_n)B + t_n B_0, \quad \widehat{h}_n = (1-t_n)h + t_n h_0.$$

After passing to a subsequence if necessary, we assume that $t_n \to t_*$ as $n \to +\infty$ with $t_* \in [0, 1]$. Then

$$\lim_{n\to+\infty} \widehat{A}_n(x) = \widehat{A}(x), \quad \lim_{n\to+\infty} \widehat{B}_n(x) = \widehat{B}(x), \quad \lim_{n\to+\infty} \widehat{h}_n(x) = \widehat{h}(x), \quad \forall\, x \in V,$$

where

$$\widehat{A}(x) = (1-t_*)A + t_* A_0 > 0, \quad \widehat{B}(x) = (1-t_*)B + t_* B_0 > 0,$$
$$\widehat{h}(x) = (1-t_*)h + t_* h_0 > 0.$$

Thus by Lemma 3.3 and Proposition 4.1, $\{u_n\}$ is bounded in $L^\infty(V)$. Contradiction arises. Therefore, the topological degree $\mathbf{d}_{h,A,B}$ is well-defined. By the homotopy invariance, we have $\mathbf{d}_{h,A,B} = \mathbf{d}_{h_0,A_0,B_0}$. We have shown in conclusion (b) that u_1 is the unique positive solution to (4.20) with $t = 1$. Hence, we conclude that

$$\mathbf{d}_{h,A,B} = \mathbf{d}_{h_0,A_0,B_0} = \operatorname{sgn} \det \big(D\mathcal{A}_{h_0,A_0,B_0}(u_1)\big) = 0.$$

In fact, applying the identity $h_0-(p-1)B_0u_1^{p-2}+(p+1)A_0u_1^{-p-2}=0$, one can check that

$$\det\left(D\mathcal{A}_{h_0,A_0,B_0}(u_1)\right)=\det\begin{pmatrix}1 & -\frac{\omega_{x_1x_2}}{\mu(x_1)} & -\frac{\omega_{x_1x_3}}{\mu(x_1)} & \cdots & -\frac{\omega_{x_1x_m}}{\mu(x_1)}\\ -\frac{\omega_{x_2x_1}}{\mu(x_2)} & 1 & -\frac{\omega_{x_2x_3}}{\mu(x_2)} & \cdots & -\frac{\omega_{x_2x_m}}{\mu(x_2)}\\ -\frac{\omega_{x_3x_1}}{\mu(x_3)} & -\frac{\omega_{x_3x_2}}{\mu(x_3)} & 1 & \cdots & -\frac{\omega_{x_3x_m}}{\mu(x_3)}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -\frac{\omega_{x_mx_1}}{\mu(x_m)} & -\frac{\omega_{x_mx_2}}{\mu(x_m)} & -\frac{\omega_{x_mx_3}}{\mu(x_m)} & \cdots & 1\end{pmatrix}=0.$$

On the other hand, we see that u_1 is a positive solution to (2.3). Therefore, by conclusion (a), we can find a positive solution $u(x)\le u_1$ on V. Consequently, the second positive solution is obtained by the fact $\mathbf{d}_{h,A,B}=0$. This finishes the proof of Theorem 2.3. □

Example 2 Let $G=(V,E)$ be a connected finite graph. Then the following *EL* equation

$$-\Delta u+\frac{2p}{p^2-1}u=\frac{1}{p-1}u^{p-1}+\frac{1}{p+1}u^{-p-1}\text{ on }V$$

possesses at least one constant solution $u(x)\equiv 1$ on V.

4.3 Mountain pass solution

In this subsection, we apply Mountain pass theorem to address the existence issue for positive solutions to Eq. (4.1), similarly to [9, Theorem 3.1]. To this end, we may consider the following "asymptotic" functional

$$\mathcal{J}_\varepsilon(u)=\frac{1}{2}\int_V\left(|\nabla u|^2+hu^2\right)\mathrm{d}\mu-\frac{1}{p}\int_V B(u^+)^p\mathrm{d}\mu+\frac{1}{p}\int_V A\left(\varepsilon+(u^+)^2\right)^{-\frac{p}{2}}\mathrm{d}\mu$$

for $\varepsilon>0$ sufficiently small and $u\in H_h^1(V)$. Naturally, Mountain pass geometry for $\mathcal{J}_\varepsilon$ could be verified and a mountain pass solution $u^{(\varepsilon)}$ will be obtained for the "asymptotic" Eq. (4.34). At this point, it remains to check that the limiting function $u=\lim\limits_{\varepsilon\to 0}u^{(\varepsilon)}$ satisfies the Eq. (4.1).

Proof of Theorem 2.4 For any fixed $\varepsilon>0$, we split $\mathcal{J}_\varepsilon$ into the sum of $\mathcal{J}^{(1)}$ and $\mathcal{J}_\varepsilon^{(2)}$, that is,

$$\mathcal{J}_\varepsilon(u)=\mathcal{J}^{(1)}(u)+\mathcal{J}_\varepsilon^{(2)}(u)$$

for $u\in H_h^1(V)$, where

$$\mathcal{J}^{(1)}(u)=\frac{1}{2}\int_V\left(|\nabla u|^2+hu^2\right)\mathrm{d}\mu-\frac{1}{p}\int_V B(x)(u^+)^p\mathrm{d}\mu$$

and

$$\mathcal{J}_\varepsilon^{(2)}(u)=\frac{1}{p}\int_V A(x)\left(\varepsilon+(u^+)^2\right)^{-\frac{p}{2}}\mathrm{d}\mu.$$

One can easily check that $\mathcal{J}_\varepsilon\in C^1\left(H_h^1(V),\mathbb{R}\right)$ by standard arguments provided $p>2$. We divide the proof into four steps.

Step 1. Mountain pass geometry. By (3.1) with $q=p$, we have

$$\left|\frac{1}{p}\int_V B(x)(u^+)^p\mathrm{d}\mu\right|\le\frac{1}{p}\max_{x\in V}B(x)\int_V|u|^p\mathrm{d}\mu\le\frac{1}{p}\max_{x\in V}B(x)S_h^p\|u\|_{H_h^1(V)}^p.$$

Then for any $u \in H_h^1(V)$, it holds that

$$G\big(\|u\|_{H_h^1(V)}\big) \le \mathcal{J}^{(1)}(u) \le F\big(\|u\|_{H_h^1(V)}\big), \tag{4.21}$$

where the functions $G(s), F(s) : [0, +\infty) \to \mathbb{R}$ are defined as

$$G(s) = \frac{s^2}{2} - \frac{1}{p}\max_{x\in V} B(x) S_h^p s^p \text{ and } F(s) = \frac{s^2}{2} + \frac{1}{p}\max_{x\in V} B(x) S_h^p s^p.$$

Let $s_0 > 0$ be such that

$$\max_{s>0} G(s) = G(s_0) = \left(\frac{1}{2} - \frac{1}{p}\right) s_0^2 \text{ with } s_0 = \left(S_h^p \max_{x\in V} B(x)\right)^{-\frac{1}{p-2}}.$$

In addition, $G(s)$ increases in $[0, s_0]$ and decreases in $[s_0, +\infty)$. Let $\kappa \in (0, 1)$ be such that $\kappa^2 = \frac{p-2}{2(p+2)}$ and set $s_1 = \kappa s_0$. Then we get

$$\begin{aligned} F(s_1) &= \kappa^2 \frac{s_0^2}{2} + \kappa^p \frac{1}{p} \max_{x\in V} B(x) S_h^p s_0^p \\ &= \kappa^2 \frac{s_0^2}{2} + \kappa^p \frac{s_0^2}{p} \le \left(\frac{1}{2} + \frac{1}{p}\right)\kappa^2 s_0^2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{p}\right) s_0^2 = \frac{1}{2} G(s_0). \end{aligned} \tag{4.22}$$

In the assumption (2.4), we set the constant $C = C(p)$ to be defined by $C(p) = \frac{p-2}{4}\kappa^p$, then (2.4) is rewritten as

$$\frac{1}{p}\int_V A(x)(s_1\zeta)^{-p} \mathrm{d}\mu \le \frac{1}{2} G(s_0). \tag{4.23}$$

Hence we deduce from (4.21), (4.22) and (4.23) that

$$\begin{cases} \mathcal{J}_\varepsilon(s_1\zeta) = \mathcal{J}^{(1)}(s_1\zeta) + \mathcal{J}_\varepsilon^{(2)}(s_1\zeta) \le F(s_1) + \dfrac{1}{p}\displaystyle\int_V A(x)(s_1\zeta)^{-p}\mathrm{d}\mu \\ \qquad \le \dfrac{1}{2}G(s_0) + \dfrac{1}{2}G(s_0) = G(s_0), \\ \mathcal{J}_\varepsilon(s_0\zeta) = \mathcal{J}^{(1)}(s_0\zeta) + \mathcal{J}_\varepsilon^{(2)}(s_0\zeta) \ge G(s_0) + \dfrac{1}{p}\displaystyle\int_V A(x)\big(\varepsilon + (s_0\zeta)^2\big)^{-\frac{p}{2}}\mathrm{d}\mu > G(s_0), \end{cases}$$

from which we conclude that

$$\mathcal{J}_\varepsilon(s_1\zeta) \le G(s_0) < \mathcal{J}_\varepsilon(s_0\zeta). \tag{4.24}$$

Noticing that for any $s > 0$, it holds

$$\mathcal{J}_\varepsilon(s\zeta) = \mathcal{J}^{(1)}(s\zeta) + \mathcal{J}_\varepsilon^{(2)}(s\zeta) = \frac{s^2}{2} - \frac{s^p}{p}\int_V B(x)\zeta^p \mathrm{d}\mu + \frac{1}{p}\int_V A(x)\big(\varepsilon + s^2\zeta^2\big)^{-\frac{p}{2}}\mathrm{d}\mu,$$

which implies that

$$\lim_{s\to+\infty} \mathcal{J}_\varepsilon(s\zeta) = -\infty.$$

Then we can choose $s_2 > s_0$ such that

$$\mathcal{J}_\varepsilon(s_2\zeta) < 0. \tag{4.25}$$

For any $u \in H_h^1(V)$ with $\|u\|_{H_h^1(V)} = s_0$, $\mathcal{J}_\varepsilon(u) > G(s_0)$ via (4.21). Therefore, the Mountain pass geometry is established on the basis of (4.24) and (4.25).

As usual, we denote by

$$c_\varepsilon := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}_\varepsilon(\gamma(t)),$$

where

$$\Gamma = \{\gamma(t) \in C([0,1]) \mid \gamma(0) = s_1\zeta,\ \gamma(1) = s_2\zeta\}.$$

Then we obtain the following two facts (i) and (ii):

(i). For any $\gamma \in \Gamma$, $\|\gamma(0)\|_{H_h^1(V)} = s_1$, $\|\gamma(1)\|_{H_h^1(V)} = s_2$, and $\|\gamma(t)\|_{H_h^1(V)}$ is continuous with respect to $t \in [0,1]$. Then for $s_0 \in (s_1, s_2)$, we deduce by the Intermediate Value Theorem that there exists some $t_0 \in (0,1)$ such that $\|\gamma(t_0)\|_{H_h^1(V)} = s_0$. Hence we have

$$\max_{t\in[0,1]} \mathcal{J}_\varepsilon(\gamma(t)) \geq \mathcal{J}_\varepsilon(\gamma(t_0)) > G(s_0) > 0,$$

which implies that

$$c_\varepsilon = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \mathcal{J}_\varepsilon(\gamma(t)) \geq G(s_0) > 0.$$

(ii). Choosing $\gamma_0(t) = \big((1-t)s_1 + ts_2\big)\zeta \in \Gamma$, then by (4.23) we have

$$\begin{aligned}\max_{t\in[0,1]} \mathcal{J}_\varepsilon(\gamma_0(t)) &= \max_{t\in[0,1]} \big(\mathcal{J}^{(1)}(\gamma_0(t)) + \mathcal{J}_\varepsilon^{(2)}(\gamma_0(t))\big)\\ &\leq \max_{s\in[s_1,s_2]} F(s) + \frac{1}{p}\int_V A(x)\big(s_1\zeta\big)^{-p}\mathrm{d}\mu\\ &= F(s_2) + \frac{1}{2}G(s_0) =: \widetilde{C} < +\infty.\end{aligned}$$

Therefore, for $\varepsilon > 0$ sufficiently small, it holds that

$$0 < G(s_0) \leq c_\varepsilon \leq \widetilde{C} < +\infty.$$

Step 2. $(\mathrm{PS})_{c_\varepsilon}$ *condition.* Let $\{u_n^{(\varepsilon)}\}_n \subseteq H_h^1(V)$ be a sequence of functions satisfying

$$\mathcal{J}_\varepsilon(u_n^{(\varepsilon)}) \to c_\varepsilon \ \text{ and } \ \|\mathcal{J}_\varepsilon'(u_n^{(\varepsilon)})\| \to 0, \ \text{ as } n \to +\infty. \tag{4.26}$$

In other words, we get

$$\begin{aligned}&\frac{1}{2}\int_V \big(|\nabla u_n^{(\varepsilon)}|^2 + h(u_n^{(\varepsilon)})^2\big)\mathrm{d}\mu - \frac{1}{p}\int_V B(x)\big((u_n^{(\varepsilon)})^+\big)^p\mathrm{d}\mu\\ &+\frac{1}{p}\int_V A(x)\Big(\varepsilon + \big((u_n^{(\varepsilon)})^+\big)^2\Big)^{-\frac{p}{2}}\mathrm{d}\mu = c_\varepsilon + o_n(1),\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}&\int_V \big(\nabla u_n^{(\varepsilon)}\cdot\nabla v + hu_n^{(\varepsilon)}v\big)\mathrm{d}\mu - \int_V B(x)\big((u_n^{(\varepsilon)})^+\big)^{p-1}v\mathrm{d}\mu\\ &= \int_V A(x)(u_n^{(\varepsilon)})^+ v\Big(\varepsilon + \big((u_n^{(\varepsilon)})^+\big)^2\Big)^{-\frac{p}{2}-1}\mathrm{d}\mu + o_n\big(\|v\|_{H_h^1(V)}\big),\end{aligned} \tag{4.28}$$

for any $v \in H_h^1(V)$. Hence by (4.27) and (4.28) with $v = u_n^{(\varepsilon)}$ we have

$$\left(\frac{1}{2} - \frac{1}{p}\right)\int_V B(x)\big((u_n^{(\varepsilon)})^+\big)^p\mathrm{d}\mu + \frac{1}{p}\int_V A(x)\Big(\varepsilon + \big((u_n^{(\varepsilon)})^+\big)^2\Big)^{-\frac{p}{2}}\mathrm{d}\mu$$

$$+\frac{1}{2}\int_V A(x)\big((u_n^{(\varepsilon)})^+\big)^2\Big(\varepsilon+\big((u_n^{(\varepsilon)})^+\big)^2\Big)^{-\frac{p}{2}-1}\mathrm{d}\mu = c_\varepsilon + o_n\big(\|u_n^{(\varepsilon)}\|_{H_h^1(V)}\big)+o_n(1). \tag{4.29}$$

So for n sufficiently large, we have

$$\frac{p-2}{2p}\int_V B(x)\big((u_n^{(\varepsilon)})^+\big)^p\mathrm{d}\mu \le 2c_\varepsilon + o_n\big(\|u_n^{(\varepsilon)}\|_{H_h^1(V)}\big). \tag{4.30}$$

Thus for n sufficiently large we deduce from (4.27) and (4.30) that

$$\int_V \big(|\nabla u_n^{(\varepsilon)}|^2 + h(u_n^{(\varepsilon)})^2\big)\mathrm{d}\mu \le \frac{4p}{p-2}c_\varepsilon + o_n\big(\|u_n^{(\varepsilon)}\|_{H_h^1(V)}\big), \tag{4.31}$$

and then

$$-2pc_\varepsilon \le \int_V B(x)\big((u_n^{(\varepsilon)})^+\big)^p\mathrm{d}\mu \le (2c_\varepsilon+1)\frac{2p}{p-2}. \tag{4.32}$$

By (4.31), we see that $\{u_n^{(\varepsilon)}\}_n$ is bounded in $H_h^1(V)$. Together with Lemma 3.1-(b), we find that $\mathcal{J}_\varepsilon$ satisfies the $(\mathrm{PS})_{c_\varepsilon}$ condition.

Step 3. Mountain pass theorem. By the Mountain pass theorem we get c_ε is a critical value. Suppose that there exists a sequence $\{u_n^{(\varepsilon)}\}_n \subseteq H_h^1(V)$ satisfying (4.26). By *Step 2*, $\{u_n^{(\varepsilon)}\}_n$ is bounded in $H_h^1(V)$. Going if necessary to a subsequence, we may assume that there exists some $u^{(\varepsilon)} \in H_h^1(V)$ such that

$$u_n^{(\varepsilon)}(x) \to u^{(\varepsilon)}(x), \quad \forall\, x\in V, \quad \text{as } n\to+\infty. \tag{4.33}$$

It follows from (4.33) and (4.28) that $u^{(\varepsilon)}$ satisfies

$$\begin{aligned}\int_V \big(\nabla u^{(\varepsilon)}\cdot\nabla v + hu^{(\varepsilon)}v\big)\mathrm{d}\mu - \int_V B(x)\big((u^{(\varepsilon)})^+\big)^{p-1}v\mathrm{d}\mu\\ = \int_V A(x)(u^{(\varepsilon)})^+v\Big(\varepsilon+\big((u^{(\varepsilon)})^+\big)^2\Big)^{-\frac{p}{2}-1}\mathrm{d}\mu,\end{aligned}$$

for any $v\in H_h^1(V)$. Choosing the test function $v=\delta_{x_0}$ for any $x_0\in V$, defined as in (2.1), we find that $u^{(\varepsilon)}$ is a point-wise solution to the following equation

$$-\Delta u^{(\varepsilon)} + hu^{(\varepsilon)} = B(x)\big((u^{(\varepsilon)})^+\big)^{p-1} + A(x)(u^{(\varepsilon)})^+\Big(\varepsilon+\big((u^{(\varepsilon)})^+\big)^2\Big)^{-\frac{p}{2}-1} \text{ on } V.$$

By Lemma 3.2-(a), $u^{(\varepsilon)}\ge 0$ on V. Consequently, $u^{(\varepsilon)}$ satisfies

$$-\Delta u^{(\varepsilon)} + hu^{(\varepsilon)} = B(x)(u^{(\varepsilon)})^{p-1} + A(x)u^{(\varepsilon)}\big(\varepsilon+(u^{(\varepsilon)})^2\big)^{-\frac{p}{2}-1} \text{ on } V. \tag{4.34}$$

Furthermore, we also conclude from Lemma 3.2-(a) that either $u^{(\varepsilon)}\equiv 0$ or $u^{(\varepsilon)}>0$ on V. Now we claim that $u^{(\varepsilon)}\equiv 0$ on V cannot happen. In fact, by (4.29) and (4.32), for n sufficiently large, we have

$$\begin{aligned}\frac{1}{p}\int_V A(x)\Big(\varepsilon+\big((u_n^{(\varepsilon)})^+\big)^2\Big)^{-\frac{p}{2}}\mathrm{d}\mu &\le -\frac{p-2}{2p}\int_V B\big((u_n^{(\varepsilon)})^+\big)^p\mathrm{d}\mu + c_\varepsilon\\ &\quad + o_n\big(\|u_n^{(\varepsilon)}\|_{H_h^1(V)}\big)+o_n(1)\\ &\le c_\varepsilon + 2pc_\varepsilon\frac{p-2}{2p} + o_n\big(\|u_n^{(\varepsilon)}\|_{H_h^1(V)}\big)+o_n(1)\\ &\le (p-1)c_\varepsilon + c_\varepsilon \le p\widetilde{C}.\end{aligned}$$

If there exists a sequence of positive numbers $\{\varepsilon_k\}_k$ with $\varepsilon_k \to 0$ as $k \to +\infty$ such that $u^{(\varepsilon_k)} \equiv 0$, then we deduce from $u_n^{(\varepsilon_k)} \to u^{(\varepsilon_k)}$ on V as $n \to +\infty$ that

$$0 < \int_V A(x)\mathrm{d}\mu \le p^2 \widetilde{C} \varepsilon_k^{\frac{p}{2}} \to 0, \quad \text{as } k \to +\infty.$$

Contradiction arises. Thus for any sufficiently small $\varepsilon > 0$, $u^{(\varepsilon)} \not\equiv 0$ on V. We conclude that $u^{(\varepsilon)}$ is a positive solution to Eq. (4.34).

Step 4. The limiting equation. By (4.31), for $\varepsilon > 0$ sufficiently small we get

$$\int_V \left(|\nabla u^{(\varepsilon)}|^2 + h(u^{(\varepsilon)})^2\right)\mathrm{d}\mu \le \frac{4p}{p-2} c_\varepsilon + c_\varepsilon \le \frac{5p-2}{p-2}\widetilde{C}. \tag{4.35}$$

Suppose that $\{\varepsilon_k\}_k$ is a sequence of positive numbers satisfying $\varepsilon_k \to 0$ as $k \to +\infty$, and $u^{(\varepsilon_k)}$ is the corresponding positive solution (obtained as in *Step 3*) to Eq. (4.34), that is, $u^{(\varepsilon_k)}$ satisfies

$$-\Delta u^{(\varepsilon_k)} + h u^{(\varepsilon_k)} = B(x)(u^{(\varepsilon_k)})^{p-1} + A(x) u^{(\varepsilon_k)} \left(\varepsilon_k + (u^{(\varepsilon_k)})^2\right)^{-\frac{p}{2}-1} \text{ on } V. \tag{4.36}$$

By (4.35), $\{u^{(\varepsilon_k)}\}_k$ is bounded in $H_h^1(V)$. Then by Lemma 3.1-(a), $\{u^{(\varepsilon_k)}\}_k$ is bounded in $L^\infty(V)$. We claim that $\{u^{(\varepsilon_k)}\}$ is uniformly bounded from below by a positive constant. Indeed, let $x_k \in V$ be such that $\min_{x\in V} u^{(\varepsilon_k)}(x) = u^{(\varepsilon_k)}(x_k)$. Then $-\Delta u^{(\varepsilon_k)}(x_k) \le 0$, and we deduce by (4.36) that

$$h(x_k) > A(x_k)\left(\varepsilon_k + (u^{(\varepsilon_k)}(x_k))^2\right)^{-\frac{p}{2}-1}. \tag{4.37}$$

Since $\varepsilon_k \to 0$ as $k \to +\infty$, for k sufficiently large, we have

$$\varepsilon_k < \frac{3}{4}\left(\min_{x\in V} \frac{A(x)}{h(x)}\right)^{\frac{2}{p+2}}.$$

Thus by (4.37), one can easily check that

$$u^{(\varepsilon_k)}(x_k) \ge \frac{1}{2}\left(\min_{x\in V}\frac{A(x)}{h(x)}\right)^{\frac{1}{p+2}} := \widehat{C} > 0.$$

Again since $\{u^{(\varepsilon_k)}\}_k$ is bounded in $H_h^1(V)$, by Lemma 3.1-(b), we may assume that there exists some $u \in H_h^1(V)$ and a subsequence, still denoted by $\{u^{(\varepsilon_k)}\}_k$, such that

$$u^{(\varepsilon_k)}(x) \to u(x), \quad \forall\, x \in V, \quad \text{as } k \to +\infty.$$

In particular, $u(x) \ge \widehat{C}$ on V. Letting $k \to +\infty$ in (4.36), we find that u satisfies (4.1). Therefore, u is a positive solution to the *EL* Eq. (4.1). This finishes the proof of Theorem 2.4. □

5 Variational analysis on the negative case

In this section, we consider the negative case

$$-\Delta u + hu = Bu^{p-1} + Au^{-p-1} \text{ on } V, \tag{5.1}$$

where $h(x) < 0$, $A(x) \ge 0$ and $A(x) \not\equiv 0$ on V unless otherwise specified.

5.1 Necessary conditions

Proposition 5.1 *(a) If* (5.1) *admits positive solutions, then it holds that*

$$\int_V B(x)\mathrm{d}\mu < 0. \tag{5.2}$$

(b) If $B(x) \geq 0$ on V, then (5.1) *admits no positive solutions.*

Proof (a) Let u be a positive solution to (5.1). Multiplying (5.1) by u^{1-p} and integrating by parts, we have

$$0 \geq \int_V \Gamma(u, u^{1-p})\mathrm{d}\mu > \int_V B(x)\mathrm{d}\mu,$$

which gives (5.2). It also implies that $B(x)$ is negative somewhere.

(b) Suppose that $B(x) \geq 0$ on V, and u is a positive solution to (5.1). Let $x_0 \in V$ be such that $u(x_0) = \min_{x\in V} u(x) > 0$. Then we have $-\Delta u(x_0) \leq 0$ and

$$0 > h(x_0) \geq B(x_0)u(x_0)^{p-2} + A(x_0)u(x_0)^{-p-2} \geq 0.$$

Contradiction arises. Of course, one can directly get conclusion (b) from (a).

□

Remark 4 For any positive solution u to Eq. (5.1), let $x_0 \in V$ be a minimum point of u. Then we have $-\Delta u(x_0) \leq 0$ and

$$0 > h(x_0) \geq B(x_0)u(x_0)^{p-2} + A(x_0)u(x_0)^{-p-2} \geq B(x_0)u(x_0)^{p-2},$$

which implies that $B(x_0) < 0$, and

$$u(x) \geq u(x_0) \geq \left(\frac{h(x_0)}{B(x_0)}\right)^{\frac{1}{p-2}}, \quad \forall\, x \in V.$$

This gives a uniform lower bound for positive solutions to (5.1), denoted by

$$u(x) \geq \left(\frac{\max_{x\in V} h(x)}{\min_{x\in V} B(x)}\right)^{\frac{1}{p-2}} > 0, \quad \forall\, x \in V. \tag{5.3}$$

Suppose that (5.2) holds and $\max_{x\in V} B(x) > 0$. Let

$$\mathcal{C}(B) = \left\{u \in W^{1,2}(V) \mid u \geq 0,\ u \not\equiv 0 \text{ on } V, \int_V B^-(x)u\mathrm{d}\mu = 0\right\}. \tag{5.4}$$

Obviously, $\mathcal{C}(B) \neq \emptyset$. In fact, $\delta_{x_0} \in \mathcal{C}(B)$, where $x_0 \in V$ is a maximum point of $B(x)$. Define

$$\lambda_B = \inf\left\{\|\nabla u\|^2_{L^2(V)}\|u\|^{-2}_{L^2(V)} \mid u \in \mathcal{C}(B)\right\}. \tag{5.5}$$

Naturally, one can check that $\lambda_B > 0$. In the rest of this section, we keep in mind that the assumptions (a), (b) and (c) in Theorem 2.5 hold.

5.2 Analysis on the energy functional

For any $\varepsilon > 0$, we consider the asymptotic equation

$$-\Delta u + hu = B(x)|u|^{p-2}u + A(x)u(u^2+\varepsilon)^{-\frac{p}{2}-1} \text{ on } V, \tag{5.6}$$

which corresponds to the energy functional $\mathcal{J}_\varepsilon(u) : W^{1,2}(V) \to \mathbb{R}$ with

$$\mathcal{J}_\varepsilon(u) = \frac{1}{2}\int_V \left(|\nabla u|^2 + hu^2\right)\mathrm{d}\mu - \frac{1}{p}\int_V B(x)|u|^p\mathrm{d}\mu + \frac{1}{p}\int_V A(x)\left(u^2+\varepsilon\right)^{-\frac{p}{2}}\mathrm{d}\mu. \tag{5.7}$$

It is easy to check that $\mathcal{J}_\varepsilon \in C^1(W^{1,2}(V), \mathbb{R})$. We aim to find the critical points of $\mathcal{J}_\varepsilon$, namely, the point-wise solutions to (5.6). To this end, we introduce the set

$$\mathcal{B}_k = \left\{u \in W^{1,2}(V) \mid u \geq 0 \text{ on } V,\ \|u\|_{L^p(V)} = k^{\frac{1}{p}}\right\},\quad \forall\, k > 0. \tag{5.8}$$

It is easy to see that $\mathcal{B}_k \neq \emptyset$, since $\overline{u}_k(x) = k^{\frac{1}{p}}|V|^{-\frac{1}{p}} \in \mathcal{B}_k$. Define

$$\Theta_k^\varepsilon := \inf_{u\in\mathcal{B}_k} \mathcal{J}_\varepsilon(u). \tag{5.9}$$

Applying the Hölder's inequality, we have

$$\frac{1}{2}\int_V hu^2\mathrm{d}\mu \geq \frac{h}{2}|V|^{1-\frac{2}{p}}k^{\frac{2}{p}},\quad \forall\, u \in \mathcal{B}_k, \tag{5.10}$$

and

$$-\frac{1}{p}\int_V B(x)|u|^p\mathrm{d}\mu \geq -\frac{k}{p}\max_{x\in V} B(x),\quad \forall\, u \in \mathcal{B}_k. \tag{5.11}$$

Combining (5.7), (5.10) and (5.11), we get that

$$\mathcal{J}_\varepsilon(u) \geq \frac{h}{2}|V|^{1-\frac{2}{p}}k^{\frac{2}{p}} - \frac{k}{p}\max_{x\in V} B(x),\quad \forall\, u \in \mathcal{B}_k. \tag{5.12}$$

Therefore by (5.9) and (5.12), $\Theta_k^\varepsilon > -\infty$ if k is finite. Furthermore, direct computation shows that

$$\Theta_k^\varepsilon \leq \mathcal{J}_\varepsilon(\overline{u}_k) = \frac{h}{2}|V|^{1-\frac{2}{p}}k^{\frac{2}{p}} - \frac{k}{p|V|}\int_V B(x)\mathrm{d}\mu + \frac{1}{p}\int_V A(x)\left(k^{\frac{2}{p}}|V|^{-\frac{2}{p}} + \varepsilon\right)^{-\frac{p}{2}}\mathrm{d}\mu, \tag{5.13}$$

which implies that $\Theta_k^\varepsilon < +\infty$. Furthermore, it is not difficult to check that $\mathcal{J}_{\varepsilon_1}(u) > \mathcal{J}_{\varepsilon_2}(u)$ for any $u \in \mathcal{B}_k$ and $\varepsilon_1 < \varepsilon_2$. This shows that Θ_k^ε is monotone decreasing with respect to ε for fixed k. Based on the above discussion, we shall study the minimization problem (5.9).

Proposition 5.2 *For fixed k and ε, Θ_k^ε is achieved by a positive function.*

Proof Suppose that $\{u_j\} \subseteq \mathcal{B}_k$ is a minimizing sequence of Θ_k^ε. Since $\{u_j\}$ is bounded in $L^p(V)$, by Lemma 3.1-(b), there exists some $u_\varepsilon^{(k)} \in V^{\mathbb{R}}$ such that up to a subsequence (still denoted by $\{u_j\}$),

$$u_j(x) \to u_\varepsilon^{(k)}(x),\quad \text{as } j \to +\infty,\quad \forall\, x \in V.$$

Then $u_\varepsilon^{(k)} \geq 0$ on V and $\|u_\varepsilon^{(k)}\|_{L^p(V)} = k^{\frac{1}{p}}$. Hence $u_\varepsilon^{(k)} \in \mathcal{B}_k$, and it is easy to check

$$\Theta_k^\varepsilon = \lim_{j\to+\infty} \mathcal{J}_\varepsilon(u_j) = \mathcal{J}_\varepsilon(u_\varepsilon^{(k)}),$$

and there exists some $\lambda \in \mathbb{R}$ such that

$$-\Delta u_\varepsilon^{(k)} + hu_\varepsilon^{(k)} = (B(x)+\lambda)(u_\varepsilon^{(k)})^{p-1} + A(x)u_\varepsilon^{(k)}\big((u_\varepsilon^{(k)})^2+\varepsilon\big)^{-\frac{p}{2}-1} \text{ on } V. \quad (5.14)$$

Since $u_\varepsilon^{(k)} \geq 0$ on V, we conclude that $u_\varepsilon^{(k)} > 0$ on V. In fact, if there exists some $x_0 \in V$ such that

$$0 = u_\varepsilon^{(k)}(x_0) = \min_{x\in V} u_\varepsilon^{(k)}(x),$$

then by (5.14), we have $\Delta u_\varepsilon^{(k)}(x_0) = 0$. Since G is connected and finite, $u_\varepsilon^{(k)}(x) \equiv u_\varepsilon^{(k)}(x_0) = 0$. This contradicts to $u_\varepsilon^{(k)} \in \mathcal{B}_k$. Therefore, $u_\varepsilon^{(k)}$ is a positive solution to (5.14). □

Now we analyze the asymptotic behavior of Θ_k^ε when both k and ε change.

Lemma 5.3 *Under the assumptions (a) and (c), it holds that*

$$\lim_{k\to0^+} \Theta_k^{k^{\frac{2}{p}}} = +\infty.$$

Furthermore, there exists some $k_\flat$ sufficiently small and independent of ε such that $\Theta_{k_\flat}^\varepsilon > 0$ for any $\varepsilon \leq k_\flat$.

Proof For any $\varepsilon \leq k^{\frac{2}{p}}$ and $u \in \mathcal{B}_k$, by Hölder's inequality and Jensen's inequality we deduce that

$$\begin{aligned}\int_V A^{\frac{1}{2}}(x)\mathrm{d}\mu &\leq \left(\int_V A(x)(u^2+\varepsilon)^{-\frac{p}{2}}\mathrm{d}\mu\right)^{\frac{1}{2}} \left(\int_V (u^2+\varepsilon)^{\frac{p}{2}}\mathrm{d}\mu\right)^{\frac{1}{2}} \\ &\leq 2^{\frac{p-2}{4}}(1+|V|)^{\frac{1}{2}} k^{\frac{1}{2}} \left(\int_V A(x)(u^2+\varepsilon)^{-\frac{p}{2}}\mathrm{d}\mu\right)^{\frac{1}{2}},\end{aligned}$$

which implies that

$$\int_V A(x)(u^2+\varepsilon)^{-\frac{p}{2}}\mathrm{d}\mu \geq \frac{1}{2^{\frac{p-2}{2}}(1+|V|)\,k}\left(\int_V A^{\frac{1}{2}}(x)\mathrm{d}\mu\right)^2. \quad (5.15)$$

Then by (5.10), (5.11) and (5.15), we obtain

$$\begin{aligned}\Theta_k^\varepsilon = \inf_{u\in\mathcal{B}_k} \mathcal{J}_\varepsilon(u) \geq \frac{h}{2}|V|^{1-\frac{2}{p}}k^{\frac{2}{p}} - \frac{k}{p}\max_{x\in V} B(x) \\ + \frac{1}{2^{\frac{p-2}{2}}p(1+|V|)\,k}\left(\int_V A^{\frac{1}{2}}(x)\mathrm{d}\mu\right)^2, \quad \forall\, u \in \mathcal{B}_k.\end{aligned}$$

Thus, we have $\Theta_k^{k^{\frac{2}{p}}} \to +\infty$ as $k \to 0^+$. One can easily choose some $k_\flat < 1$ independent of ε such that

$$\frac{h}{2}|V|^{1-\frac{2}{p}}k_\flat^{\frac{2}{p}} - \frac{k_\flat}{p}\max_{x\in V} B(x) + \frac{1}{2^{\frac{p-2}{2}}p(1+|V|)\,k_\flat}\left(\int_V A^{\frac{1}{2}}(x)\mathrm{d}\mu\right)^2 > 0.$$

In fact, we can choose $k_\flat$ as

$$k_\flat = \min\left\{ \frac{\left(\int_V A^{\frac{1}{2}}(x)\mathrm{d}\mu\right)^2}{2^{\frac{p}{2}-2} p\,(1+|V|)\left(|h||V|^{1-\frac{2}{p}} + \frac{2}{p}\max_{x\in V} B(x)\right)}, \left(\frac{|h||V|^{2-\frac{2}{p}}}{\int_V B^-(x)\mathrm{d}\mu}\right)^{\frac{p}{p-2}}, 1 \right\}. \tag{5.16}$$

Here we have used the fact that $k_\flat < k_\flat^{\frac{2}{p}}$. As a consequence, we have

$$0 < \Theta_{k_\flat}^{k_\flat^{\frac{2}{p}}} \le \Theta_{k_\flat}^{k_\flat} \le \Theta_{k_\flat}^{\varepsilon}, \text{ for any } \varepsilon \le k_\flat.$$

This finishes the proof of Lemma 5.3. □

Next we analyze the asymptotic behavior of Θ_k^ε as $k \to +\infty$.

Lemma 5.4 *Under the assumptions (a) and (c), for any fixed $\varepsilon > 0$, it holds that*

$$\lim_{k\to+\infty} \Theta_k^\varepsilon = -\infty.$$

Proof Let $\Omega = \{x \in V \mid B(x) > 0\}$ and define

$$\chi_\Omega(x) = \begin{cases} 1, & \text{if } x \in \Omega, \\ 0, & \text{if } x \in V \setminus \Omega. \end{cases}$$

Obviously, $\Omega \ne \emptyset$ which is due to the assumption (c) in Theorem 2.5. Denote by

$$\zeta(t) = \int_V B(x)e^{t\chi_\Omega(x)}\mathrm{d}\mu = \sum_{x\in\Omega} \mu(x)B(x)e^t + \sum_{x\in V\setminus\Omega} \mu(x)B(x), \quad t \in \mathbb{R}. \tag{5.17}$$

Then $\zeta(t)$ is smooth in $\mathbb{R}$ and $\zeta(0) < 0$ by the assumption (5.2). For any $t \in \mathbb{R}$, we have

$$\begin{aligned} \zeta(t) &= \int_V B^+(x)e^{t\chi_\Omega(x)}\mathrm{d}\mu - \int_{V\setminus\Omega} B^-(x)e^{t\chi_\Omega(x)}\mathrm{d}\mu \\ &= \int_\Omega B^+(x)e^t\mathrm{d}\mu - \int_{V\setminus\Omega} B^-(x)\mathrm{d}\mu \\ &\ge \min_{x\in V} B^+(x)|\Omega|e^t - \int_V B^-(x)\mathrm{d}\mu. \end{aligned}$$

Thus, there exists some $t_0 \gg 1$ such that $\zeta(t_0) \ge 1$. Moreover, direct computation shows that

$$\zeta'(t) = \sum_{x\in\Omega} \mu(x)B(x)e^t = \int_V B(x)\chi_\Omega(x)e^{t\chi_\Omega(x)}\mathrm{d}\mu > 0,$$

which implies that $\zeta(t)$ is strictly increasing in $\mathbb{R}$ and $\zeta(t) \ge 1$ for any $t \ge t_0$.

We choose a positive function $v(x) = c\exp\left(t_0\chi_\Omega(x)\right)$, $x \in V$, where $c > 0$ is determined by

$$|V| = \int_V v^p\mathrm{d}\mu = c^p \int_V e^{pt_0\chi_\Omega(x)}\mathrm{d}\mu. \tag{5.18}$$

Thus, it is easy to see that

$$\int_V B(x)v^p \mathrm{d}\mu = c^p \int_V B(x)e^{pt_0\chi_\Omega(x)}\mathrm{d}\mu = c^p\zeta(pt_0) > c^p\zeta(t_0) > 0. \tag{5.19}$$

Since $\overline{u}_k v = k^{\frac{1}{p}}|V|^{-\frac{1}{p}}v \in \mathcal{B}_k$, we have

$$\begin{aligned}\mathcal{J}_\varepsilon(\overline{u}_k v) &= \frac{1}{2}k^{\frac{2}{p}}|V|^{-\frac{2}{p}}\int_V \left(|\nabla v|^2 + hv^2\right)\mathrm{d}\mu - \frac{k}{p|V|}\int_V B(x)v^p\mathrm{d}\mu \\ &\quad + \frac{1}{p}\int_V A(x)\left(k^{\frac{2}{p}}|V|^{-\frac{2}{p}}v^2 + \varepsilon\right)^{-\frac{p}{2}}\mathrm{d}\mu \\ &\le \frac{1}{2}k^{\frac{2}{p}}|V|^{-\frac{2}{p}}\int_V \left(|\nabla v|^2 + hv^2\right)\mathrm{d}\mu - \frac{k}{p|V|}\int_V B(x)v^p\mathrm{d}\mu + \frac{1}{p}\varepsilon^{-\frac{p}{2}}\int_V A(x)\mathrm{d}\mu,\end{aligned}$$

which shows that $\mathcal{J}_\varepsilon(\overline{u}_k v) \to -\infty$ as $k \to +\infty$ by (5.19). This finishes the proof of Lemma 5.4. □

Lemma 5.5 *Under the assumptions (a), (b) and (c), there exists some $k_\natural > 0$ independent of ε such that $\Theta^\varepsilon_{k_\natural} \le 0$ for any $\varepsilon > 0$. In particular, $k_\natural > k_\flat$.*

Proof By (5.13), we have

$$\mathcal{J}_\varepsilon(\overline{u}_k) \le \frac{h}{2}|V|^{1-\frac{2}{p}}k^{\frac{2}{p}} + \frac{k}{p|V|}\int_V B^-(x)\mathrm{d}\mu + \frac{|V|}{pk}\int_V A(x)\mathrm{d}\mu. \tag{5.20}$$

The right hand side of (5.20) is non-positive if and only if

$$\int_V A(x)\mathrm{d}\mu \le \frac{|h|p}{2|V|^{\frac{2}{p}}}k^{\frac{2}{p}+1} - \frac{k^2}{|V|^2}\int_V B^-(x)\mathrm{d}\mu =: \eta(k).$$

One can easily check that

$$\max_{k>0}\eta(k) = \eta(k_\natural) = \left(\frac{p+2}{4}\frac{|h|}{\int_V B^-(x)\mathrm{d}\mu}\right)^{\frac{p+2}{p-2}}\frac{|h|(p-2)|V|^{\frac{2p}{p-2}}}{4},$$

where

$$k_\natural = \left(\frac{p+2}{4}\frac{|h||V|^{2-\frac{2}{p}}}{\int_V B^-(x)\mathrm{d}\mu}\right)^{\frac{p}{p-2}}. \tag{5.21}$$

Hence by the assumption (2.5), we deduce that $\Theta^\varepsilon_{k_\natural} \le \mathcal{J}_\varepsilon(\overline{u}_k) \le 0$ for any $\varepsilon > 0$. Furthermore, by (5.16) and the fact $p > 2$, we have

$$k_\natural > \min\left\{\left(\frac{|h||V|^{2-\frac{2}{p}}}{\int_V B^-(x)\mathrm{d}\mu}\right)^{\frac{p}{p-2}}, 1\right\} \ge k_\flat.$$

This finishes the proof of Lemma 5.5. □

Remark 5 By (5.13) and (5.20), we further obtain that

$$\mathcal{J}_\varepsilon(\overline{u}_{k_\natural}) \le -\frac{k_\natural}{p|V|}\int_V B^+(x)\mathrm{d}\mu.$$

Since $\max_{x\in V} B(x) > 0$, we get

$$\Theta^{\varepsilon}_{k_\natural} \le -\frac{1}{p|V|}\min\left\{\left(\frac{|h||V|^{2-\frac{2}{p}}}{\int_V B^-(x)\mathrm{d}\mu}\right)^{\frac{p}{p-2}}, 1\right\}\int_V B^+(x)\mathrm{d}\mu < 0, \quad \text{for any } \varepsilon > 0.$$

Proposition 5.6 *Suppose that the assumptions (a), (b) and (c) hold.*

(a) There exists some constant Θ *independent of* ε *such that* $\Theta^{\varepsilon}_k \le \Theta$ *for any* $\varepsilon > 0$ *and* $k \ge k_\flat$.
(b) There exists some $k_\sharp$ *sufficiently large and independent of* ε *such that* $\Theta^{\varepsilon}_k < 0$ *for any* $k \ge k_\sharp$.

Proof (a) By the same arguments of the proof of Lemma 5.4, we can define a positive function $v(x) = ce^{t_0\chi_\Omega(x)}$, $x \in V$, where $c > 0$ satisfies (5.18). Naturally, we have $v(x) \ge c$ for any $x \in V$ and (5.19) holds. Since $\overline{u}_k v = k^{\frac{1}{p}}|V|^{-\frac{1}{p}}v \in \mathcal{B}_k$ and $h < 0$, we deduce by (5.7) that

$$\mathcal{J}_\varepsilon(\overline{u}_k v) \le \frac{1}{2}k^{\frac{2}{p}}|V|^{-\frac{2}{p}}\int_V |\nabla v|^2\mathrm{d}\mu - \frac{k}{p|V|}\int_V B(x)v^p\mathrm{d}\mu + \frac{|V|}{pc^pk}\int_V A(x)\mathrm{d}\mu. \tag{5.22}$$

As a function of k, the right hand side of (5.22) achieves its maximum for $k \ge k_\flat$, denoted by Θ.

(b) The right hand side of (5.22), being considered as a function of k, is continuous and independent of ε. We know that the function on the right hand side of (5.22) goes to $-\infty$ as $k \to +\infty$. Hence we can choose $k_\sharp > \max\{k_\natural, 1\}$ sufficiently large such that $\Theta^{\varepsilon}_k < 0$ for any $k \ge k_\sharp$. This completes the proof of Proposition 5.6.

□

Lemma 5.7 *For any fixed* $\varepsilon > 0$, Θ^{ε}_k *is continuous with respect to* k.

Proof By (5.12) and (5.13), we see that Θ^{ε}_k is well-defined for any $k > 0$. Now we need to check that for any $k > 0$ and any sequence $\{k_j\}$ with $k_j \to k$ as $j \to +\infty$, it holds that

$$\lim_{j\to+\infty}\Theta^{\varepsilon}_{k_j} = \Theta^{\varepsilon}_k. \tag{5.23}$$

By Proposition 5.2, we suppose that Θ^{ε}_k and $\Theta^{\varepsilon}_{k_j}$ are achieved by $u \in \mathcal{B}_k$ and $u_j \in \mathcal{B}_{k_j}$ respectively, furthermore, u and u_j are positive on V.

Next we choose a sequence of positive numbers $\{t_j\}$ such that $t_ju \in \mathcal{B}_{k_j}$. Then we have $k_j^{\frac{1}{p}} = \|t_ju\|_{L^p(V)} = t_jk^{\frac{1}{p}}$, which implies that $t_j \to 1$ as $j \to +\infty$. Therefore, we get

$$\Theta^{\varepsilon}_{k_j} \le \mathcal{J}_\varepsilon(t_ju) = \frac{t_j^2}{2}\int_V \left(|\nabla u|^2 + hu^2\right)\mathrm{d}\mu - \frac{t_j^p}{p}\int_V B(x)u^p\mathrm{d}\mu$$
$$+\frac{1}{p}\int_V A(x)\left(t_j^2u^2 + \varepsilon\right)^{-\frac{p}{2}}\mathrm{d}\mu,$$

which yields that

$$\limsup_{j\to+\infty}\Theta^{\varepsilon}_{k_j} \le \mathcal{J}_\varepsilon(u). \tag{5.24}$$

On the other hand, since $\|u_j\|_{L^p(V)} = k_j^{\frac{1}{p}} \to k^{\frac{1}{p}}$ as $j \to +\infty$, by Lemma 3.1-(b), there exists some $\widetilde{u} \in V^{\mathbb{R}}$ such that up to a subsequence (still denoted by $\{u_j\}$), $u_j(x) \to \widetilde{u}(x)$ as $j \to +\infty$ for any $x \in V$. Thus, it holds that $\|\widetilde{u}\|_{L^p(V)} = \lim_{j\to+\infty} \|u_j\|_{L^p(V)} = k^{\frac{1}{p}}$ and then $\widetilde{u} \in \mathcal{B}_k$. Therefore,

$$\mathcal{J}_\varepsilon(u) \le \mathcal{J}_\varepsilon(\widetilde{u}) = \lim_{j\to+\infty} \mathcal{J}_\varepsilon(u_j) = \lim_{j\to+\infty} \Theta^\varepsilon_{k_j}. \tag{5.25}$$

Combining (5.24) and (5.25), we conclude that $\lim_{j\to+\infty} \Theta^\varepsilon_{k_j} = \mathcal{J}_\varepsilon(u) = \Theta^\varepsilon_k$. This gives (5.23).

□

5.3 Proof of Theorem 2.5

Before giving the proof of Theorem 2.5, we first introduce the quantity $\lambda_{B,\gamma}$.

Lemma 5.8 *Suppose that the assumption (c) holds. For any $\gamma > 0$, we set*

$$\lambda_{B,\gamma} = \inf\left\{\|\nabla u\|^2_{L^2(V)}\|u\|^{-2}_{L^2(V)} \mid u \in \mathcal{C}(B,\gamma)\right\}, \tag{5.26}$$

where

$$\begin{aligned}\mathcal{C}(B,\gamma) = \Big\{u \in W^{1,2}(V) \mid u \ge 0 \text{ on } V,\ \|u\|_{L^p(V)} = 1,\ \int_V B^-(x)u^p \mathrm{d}\mu \\ = \gamma \int_V B^-(x)\mathrm{d}\mu\Big\}.\end{aligned} \tag{5.27}$$

Then $\lambda_{B,\gamma}$ is monotone decreasing with respect to γ.

Proof We split the proof into three steps.
Step 1. We first consider the following minimizing problem

$$\lambda'_{B,\gamma} = \inf\left\{\|\nabla u\|^2_{L^2(V)}\|u\|^{-2}_{L^2(V)} \mid u \in \mathcal{C}'(B,\gamma)\right\},$$

where

$$\begin{aligned}\mathcal{C}'(B,\gamma) = \Big\{u \in W^{1,2}(V) \mid u \ge 0 \text{ on } V,\ \|u\|_{L^p(V)} = 1, \\ \int_V B^-(x)u^p \mathrm{d}\mu \le \gamma \int_V B^-(x)\mathrm{d}\mu\Big\}.\end{aligned}$$

Choose any $x_0 \in \Omega := \{x \in V \mid B(x) > 0\}$ and denote $u_\gamma(x) = \mu(x_0)^{-\frac{1}{p}}\delta_{x_0}$. Then $u_\gamma \in \mathcal{C}'(B,\gamma)$. This implies that $\mathcal{C}'(B,\gamma) \ne \emptyset$ and thus $\lambda'_{B,\gamma}$ is finite. By the definition of $\mathcal{C}'(B,\gamma)$, if $\gamma_1 \le \gamma_2$, then $\mathcal{C}'(B,\gamma_1) \subseteq \mathcal{C}'(B,\gamma_2)$, and hence $\lambda'_{B,\gamma_2} \le \lambda'_{B,\gamma_1}$. This yields that $\lambda'_{B,\gamma}$ is monotone decreasing with respect to γ. In the sequel, we shall prove that $\lambda'_{B,\gamma} = \lambda_{B,\gamma}$. By the fact that $\mathcal{C}(B,\gamma) \subseteq \mathcal{C}'(B,\gamma)$, we find $\lambda'_{B,\gamma} \le \lambda_{B,\gamma}$. So it remains to check that $\lambda'_{B,\gamma} \ge \lambda_{B,\gamma}$.
Step 2. $\lambda'_{B,\gamma}$ is achieved by some $u \in \mathcal{C}(B,\gamma)$. In fact, suppose that $\{u_j\} \subseteq \mathcal{C}'(B,\gamma)$ is a minimizing sequence of $\lambda'_{B,\gamma}$, then $\{u_j\}$ is nonnegative and bounded in $L^p(V)$. By Lemma 3.1-(b), we assume that up to a subsequence (still denoted by $\{u_j\}$),

$$u_j(x) \to u(x), \quad \text{as } j \to +\infty, \quad \forall\, x \in V.$$

Thus one can check that $u \in \mathcal{C}'(B,\gamma)$, and then $\lambda'_{B,\gamma}$ is achieved by u. We claim that $u \in \mathcal{C}(B,\gamma)$. Otherwise, there exists some constant $\kappa > 0$ such that

$$\int_V B^-(x)(u+\kappa)^p \mathrm{d}\mu = \gamma \int_V B^-(x)\mathrm{d}\mu.$$

Since $(u+\kappa)\|u+\kappa\|^{-1}_{L^p(V)} \in \mathcal{C}'(B,\gamma)$, we have

$$\left\|\nabla\left(\frac{u+\kappa}{\|u+\kappa\|_{L^2(V)}}\right)\right\|^2_{L^2(V)} \left\|\frac{u+\kappa}{\|u+\kappa\|_{L^2(V)}}\right\|^{-2}_{L^2(V)} = \frac{\|\nabla(u+\kappa)\|^2_{L^2(V)}}{\|u+\kappa\|^2_{L^2(V)}} < \frac{\|\nabla u\|^2_{L^2(V)}}{\|u\|^2_{L^2(V)}}.$$

Contradiction arises. Hence $u \in \mathcal{C}(B,\gamma)$. This also yields that $\mathcal{C}(B,\gamma) \neq \emptyset$.
Step 3. By the definition of $\lambda_{B,\gamma}$, we conclude that

$$\lambda'_{B,\gamma} = \|\nabla u\|^2_{L^2(V)}\|u\|^{-2}_{L^2(V)} \geq \lambda_{B,\gamma}.$$

This finishes the proof of Lemma 5.8. In addition, $\lambda_{B,\gamma}$ is achieved by the u as well. □

Lemma 5.9 *Suppose that the assumption (c) holds. Then $\lambda_{B,\gamma} \leq \lambda_B$ for any $\gamma > 0$. In particular,*

$$\lim_{\gamma\to 0^+} \lambda_{B,\gamma} = \lambda_B.$$

Proof For any $u \in \mathcal{C}(B)$, by (5.4) and the assumption (c), we have

$$\int_V u^p \mathrm{d}\mu > 0, \quad \int_V B^-(x)u^p \mathrm{d}\mu = 0.$$

We choose some constant $c > 0$ such that $\|cu\|_{L^p(V)} = 1$. This implies that $cu \in \mathcal{C}'(B,\gamma)$. Therefore, we get that

$$\lambda'_{B,\gamma} \leq \|\nabla(cu)\|^2_{L^2(V)}\|cu\|^{-2}_{L^2(V)} = \|\nabla u\|^2_{L^2(V)}\|u\|^{-2}_{L^2(V)}.$$

Taking the infimum with respect to u on both sides over the set $\mathcal{C}(B)$, we have $\lambda'_{B,\gamma} \leq \lambda_B$. Together with the fact $\lambda'_{B,\gamma} = \lambda_{B,\gamma}$ we get $\lambda_{B,\gamma} \leq \lambda_B$.

Next we shall verify that $\lambda_B = \lim_{\gamma\to 0^+}\lambda_{B,\gamma}$. Suppose this is not true, then there is some $\varepsilon_0 > 0$ such that for any $\gamma_0 > 0$, there exists $\gamma < \gamma_0$ satisfying $\lambda_B - \lambda_{B,\gamma} \geq \varepsilon_0$. If $\gamma_0 \to 0^+$, then $\gamma \to 0^+$. For this sequence $\{\lambda_{B,\gamma}\}_\gamma$, we assume that $\lambda_{B,\gamma}$ is achieved by $v_\gamma \in \mathcal{C}(B,\gamma)$. Thus we get

$$\|\nabla v_\gamma\|^2_{L^2(V)}\|v_\gamma\|^{-2}_{L^2(V)} \leq \lambda_{B,\gamma} \leq \lambda_B - \varepsilon_0.$$

Using $\|v_\gamma\|_{L^p(V)} = 1$ and the above inequality, we find that $\{v_\gamma\}_\gamma$ is bounded in $W^{1,2}(V)$. Then by Lemma 3.1-(b), up to a subsequence (still denoted by $\{v_\gamma\}_\gamma$), there exists $v \in W^{1,2}(V)$ such that $v_\gamma(x) \to v(x)$ as $\gamma \to 0^+$, for any $x \in V$. Thus we have

$$\int_V v^p \mathrm{d}\mu = \lim_{\gamma\to 0^+}\int_V v_\gamma^p \mathrm{d}\mu = 1,$$

and

$$\int_V B^-(x)v^p \mathrm{d}\mu = \lim_{\gamma\to 0^+}\int_V B^-(x)v_\gamma^p \mathrm{d}\mu = \lim_{\gamma\to 0^+}\gamma\int_V B^-(x)\mathrm{d}\mu = 0.$$

This implies that $v \in \mathcal{C}(B)$. Therefore, we have

$$\lambda_B \le \|\nabla v\|^2_{L^2(V)}\|v\|^{-2}_{L^2(V)} = \lim_{\gamma\to 0^+} \|\nabla v_\gamma\|^2_{L^2(V)}\|v_\gamma\|^{-2}_{L^2(V)} \le \lambda_B - \varepsilon_0.$$

Contradiction arises. This finishes the proof of Lemma 5.9. □

Lemma 5.10 *Suppose that the assumptions (a), (b) and (c) hold. Then there exists $\gamma_0 \in (0, \frac{4p}{(p+2)|V|})$ sufficiently small such that*

$$\delta := \frac{\lambda_{B,\gamma_0} + h}{2} > \frac{3}{8}(\lambda_B + h). \tag{5.28}$$

For this δ, we denote by

$$\Upsilon_1 = \min\left\{\frac{|h||V|^{1-\frac{2}{p}}}{2},\ \frac{\delta}{S_1^2(4\delta + 2|h| + 1)}\right\}, \tag{5.29}$$

where S_1 is the Sobolev embedding constant (see (3.1) with $h(x) \equiv 1$ and $q = p$), and set

$$\Upsilon_2 := \frac{\gamma_0}{4|h||V|^{1-\frac{2}{p}}}\Upsilon_1. \tag{5.30}$$

If

$$\max_{x\in V} B(x) < \Upsilon_2 \int_V B^-(x)\mathrm{d}\mu, \tag{5.31}$$

then there exists some k_* independent of ε such that $\mathcal{J}_\varepsilon(u) > \frac{1}{2}\Upsilon_1 k_*^{\frac{2}{p}}$ for any $\varepsilon > 0$ and $u \in \mathcal{B}_{k_*}$. In particular, $\Theta^\varepsilon_{k_*} > 0$ for any $\varepsilon > 0$, and $k_\natural < k_* < k_\sharp$.

Proof By Lemma 5.9, there exists some $\gamma_0 \in (0, \frac{4p}{(p+2)|V|})$ small enough such that

$$0 \le \lambda_B - \lambda_{B,\gamma_0} < \frac{1}{4}(\lambda_B - |h|).$$

This gives (5.28). We set

$$k_* = \left(\frac{p|h||V|^{1-\frac{2}{p}}}{\gamma_0\int_V B^-(x)\mathrm{d}\mu}\right)^{\frac{p}{p-2}}. \tag{5.32}$$

It is easy to check that $k_* > k_\natural$ by (5.21). Now we assume that $k \ge k_*$, and decompose $\mathcal{J}_\varepsilon$ as

$$\mathcal{J}_\varepsilon(u) = \mathcal{G}(u) - \frac{1}{p}\int_V B^+(x)|u|^p\mathrm{d}\mu + \frac{1}{p}\int_V A(x)\left(u^2+\varepsilon\right)^{-\frac{p}{2}}\mathrm{d}\mu, \tag{5.33}$$

where

$$\mathcal{G}(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(V)} + \frac{h}{2}\|u\|^2_{L^2(V)} + \frac{1}{p}\int_V B^-(x)|u|^p\mathrm{d}\mu. \tag{5.34}$$

For any $u \in \mathcal{B}_k$, we consider the following two cases:
Case 1. Suppose that

$$\int_V B^-(x)u^p\mathrm{d}\mu \ge \gamma_0 k\int_V B^-(x)\mathrm{d}\mu.$$

Since $k \geq k_*$, by the Hölder's inequality (see (5.10)) and (5.32), we have

$$\begin{aligned} \mathcal{G}(u) &\geq \frac{h}{2}\|u\|^2_{L^2(V)} + \frac{\gamma_0 k}{p}\int_V B^-(x)\mathrm{d}\mu \\ &\geq \frac{|h|}{2}k^{\frac{2}{p}}\left(\frac{2\gamma_0\int_V B^-(x)\mathrm{d}\mu}{|h|p}k^{1-\frac{2}{p}} - |V|^{1-\frac{2}{p}}\right) \\ &\geq \frac{|h|}{2}|V|^{1-\frac{2}{p}}k^{\frac{2}{p}}. \end{aligned} \tag{5.35}$$

Case 2. Suppose that

$$\int_V B^-(x)u^p\mathrm{d}\mu < \gamma_0 k\int_V B^-(x)\mathrm{d}\mu.$$

Since $u \in \mathcal{B}_k$, $k^{-\frac{1}{p}}u \in \mathcal{C}'(B,\gamma_0)$. By the proof of Lemma 5.8 and the definition of λ_{B,γ_0}, it holds

$$\lambda_{B,\gamma_0} = \lambda'_{B,\gamma_0} \leq \|\nabla(k^{-\frac{1}{p}}u)\|^2_{L^2(V)}\|k^{-\frac{1}{p}}u\|^{-2}_{L^2(V)} = \|\nabla u\|^2_{L^2(V)}\|u\|^{-2}_{L^2(V)}. \tag{5.36}$$

We rewrite (5.34) as

$$\|u\|^2_{L^2(V)} = \frac{2}{|h|}\left(\frac{1}{2}\|\nabla u\|^2_{L^2(V)} + \frac{1}{p}\int_V B^-(x)u^p\mathrm{d}\mu - \mathcal{G}(u)\right). \tag{5.37}$$

Therefore, by (5.28) and (5.36), we get

$$\begin{aligned} \mathcal{G}(u) &\geq \frac{1}{2}(\lambda_{B,\gamma_0}+h)\|u\|^2_{L^2(V)} + \frac{1}{p}\int_V B^-(x)u^p\mathrm{d}\mu \\ &\geq \delta\|u\|^2_{L^2(V)} = \frac{\delta}{2|h|+1}\|u\|^2_{L^2(V)} + \frac{2|h|\delta}{2|h|+1}\|u\|^2_{L^2(V)} \\ &\geq \frac{\delta}{2|h|+1}\|u\|^2_{L^2(V)} + \frac{4\delta}{(2|h|+1)}\left(\frac{1}{2}\|\nabla u\|^2_{L^2(V)} - \mathcal{G}(u)\right). \end{aligned} \tag{5.38}$$

We solve inequality (5.38) with respect to $\mathcal{G}(u)$, and then deduce by (3.1) that

$$\begin{aligned} \mathcal{G}(u) &\geq \frac{\delta}{4\delta+2|h|+1}\left(\|u\|^2_{L^2(V)} + 2\|\nabla u\|^2_{L^2(V)}\right) \geq \frac{\delta}{4\delta+2|h|+1}\|u\|^2_{W^{1,2}(V)} \\ &\geq \frac{\delta}{S_1^2(4\delta+2|h|+1)}\|u\|^2_{L^p(V)} = \frac{\delta}{S_1^2(4\delta+2|h|+1)}k^{\frac{2}{p}}. \end{aligned} \tag{5.39}$$

In any case, it follows from (5.29), (5.35) and (5.39) that $\mathcal{G}(u) \geq \Upsilon_1 k^{\frac{2}{p}}$. Thus, we deduce that for any $u \in \mathcal{B}_k$ and any $k \geq k_*$,

$$\mathcal{J}_\varepsilon(u) \geq \Upsilon_1 k^{\frac{2}{p}} - \frac{k}{p}\max_{x\in V} B(x).$$

Applying (5.30) and (5.31), one can easily check that $\left(\frac{p\Upsilon_1}{2\max_{x\in V}B(x)}\right)^{\frac{p}{p-2}} > k_*$. Then for any $u \in \mathcal{B}_{k_*}$, we have $\mathcal{J}_\varepsilon(u) > \frac{1}{2}\Upsilon_1 k_*^{\frac{2}{p}}$. Therefore, $\Theta^\varepsilon_{k_*} > 0$ for any $\varepsilon > 0$. By Proposition 5.6-(b), we see $k_* < k_\natural$. This finishes the proof of Lemma 5.10. □

Proof of Theorem 2.5 For sufficiently small $\varepsilon > 0$, there will be $k_\natural$ and $k_\flat$ such that $k_\natural > k_\flat$ and $\Theta^\varepsilon_{k_\natural} \leq 0$ while $\Theta^\varepsilon_{k_\flat} > 0$. In fact, by Lemma 5.3, there exists some $k_\flat > 0$ independent of ε such that $\Theta^{\varepsilon}_{k_\flat} > 0$ for any $\varepsilon \le k_\flat$. By Lemma 5.5, there exists some $k_\natural > 0$ independent of ε such that $\Theta^{\varepsilon}_{k_\natural} \le 0$ for any $\varepsilon > 0$. In addition, $k_\natural > k_\flat$. In the sequel, we assume $\varepsilon \le k_\flat$.

Let k_* be defined as in Lemma 5.10 satisfying that $\Theta^{\varepsilon}_{k_*} > 0$ for any $\varepsilon > 0$. Fix $\varepsilon > 0$ and we consider the minimizing problem

$$\widehat{\Theta}^{\varepsilon} := \inf_{u \in \mathcal{D}_k} \mathcal{J}_\varepsilon(u) = \min_{k_\flat \le k \le k_*} \Theta^{\varepsilon}_k,$$

where

$$\mathcal{D}_k = \left\{ u \in W^{1,2}(V) \mid u \ge 0 \text{ on } V,\ k_\flat \le \|u\|^p_{L^p(V)} \le k_* \right\}.$$

For any $k \in [k_\flat, k_*]$ and any $\varepsilon > 0$, by (5.12) and Proposition 5.6-(a), we see that Θ^{ε}_k is uniformly bounded, thus $\widehat{\Theta}^{\varepsilon}$ is well-defined. By Proposition 5.2, each Θ^{ε}_k is achieved by a positive function. Therefore, by Lemma 5.7, we conclude that $\widehat{\Theta}^{\varepsilon}$ is achieved by a positive function, denoted by $u^{(\varepsilon)} \in \mathcal{D}_k$. In another way, similar to the arguments of Proposition 5.2, one can directly prove that $\widehat{\Theta}^{\varepsilon}$ is achieved by some positive function $u^{(\varepsilon)} \in \mathcal{D}_k$.

By Lemma 5.5 and Remark 5, we find that

$$\widehat{\Theta}^{\varepsilon} \le \Theta^{\varepsilon}_{k_\natural} < 0. \tag{5.40}$$

Namely, the energy $\widehat{\Theta}^{\varepsilon} = \mathcal{J}_\varepsilon(u^{(\varepsilon)})$ is strictly negative. Let $\|u^{(\varepsilon)}\|^p_{L^p(V)} = k^{(\varepsilon)} \in [k_\flat, k_*]$. Notice that both $\Theta^{\varepsilon}_{k_\flat}$ and $\Theta^{\varepsilon}_{k_*}$ are positive, we see $k^{(\varepsilon)} \in (k_\flat, k_*)$. In addition, $u^{(\varepsilon)}$ is a positive solution to the asymptotic Eq. (5.6).

Since $\{u^{(\varepsilon)}\}_\varepsilon$ is bounded in $L^p(V)$ by the constant $k_*^{\frac{1}{p}}$, hence bounded in $L^\infty(V)$, one can easily deduce by Lemma 3.1-(b) that up to a subsequence, denoted by $\{u^{(\varepsilon_j)}\}_j$, we have

$$u^{(\varepsilon_j)}(x) \to u(x), \quad \text{as } j \to +\infty, \quad \forall\, x \in V.$$

Here $u \in L^\infty(V)$ is nonnegative. Since $u^{(\varepsilon_j)}$ is a positive solution to (5.6) with $\varepsilon = \varepsilon_j$, by the same arguments of Remark 4, we conclude that $\{u^{(\varepsilon_j)}\}_j$ is uniformly bounded from below. Thus, u is positive on V. Letting $j \to +\infty$ in (5.6) with $\varepsilon = \varepsilon_j$, we conclude that u is a positive solution to (5.1), and the energy is strictly negative by (5.40). This finishes the proof of Theorem 2.5. □

6 Heat flow and topological degree for the null case

In the last section, we address the null case

$$-\Delta u = Bu^{p-1} + Au^{-p-1} \text{ on } V, \tag{6.1}$$

where $p > 2$, $A(x) > 0$ and $B(x) < 0$ on V unless otherwise specified.

Remark 6 Suppose that u is a positive solution to (6.1). Letting $x_0 \in V$ be a minimum point of u on V, then we get

$$0 \ge -\Delta u(x_0) = B(x_0)u(x_0)^{p-1} + A(x_0)u(x_0)^{-p-1},$$

which implies that

$$u(x) \ge u(x_0) \ge \left(\frac{A(x_0)}{-B(x_0)}\right)^{\frac{1}{2p}} \ge \left(\min_{x \in V} \frac{A(x)}{-B(x)}\right)^{\frac{1}{2p}}, \quad \forall\, x \in V. \tag{6.2}$$

Similarly, we denote by $u(x_1) = \max_{x\in V} u(x)$ and then have

$$0 \le -\Delta u(x_1) = B(x_1)u(x_1)^{p-1} + A(x_1)u(x_1)^{-p-1},$$

which yields that

$$u(x) \le u(x_1) \le \left(\frac{A(x_1)}{-B(x_1)}\right)^{\frac{1}{2p}} \le \left(\max_{x\in V}\frac{A(x)}{-B(x)}\right)^{\frac{1}{2p}}, \quad \forall\, x \in V. \tag{6.3}$$

Hence by (6.2) and (6.3), we obtain the a-priori estimate for the positive solutions to (6.1) as the following

$$\underline{u}(x) \equiv \left(\min_{x\in V}\frac{A(x)}{-B(x)}\right)^{\frac{1}{2p}} \le u(x) \le \left(\max_{x\in V}\frac{A(x)}{-B(x)}\right)^{\frac{1}{2p}} \equiv \overline{u}(x), \quad \forall\, x \in V. \tag{6.4}$$

One can easily check that $\underline{u}$ and $\overline{u}$ are sub- and super-solutions to (6.1) respectively. Therefore, we shall apply the sub- and super-solution method (similar to Theorem 2.2) to find solutions to (6.1). Namely, we can find some minimizer $\widetilde{u}$ of the energy functional $\mathcal{J}$ in $\mathcal{N} = \{u \in W^{1,2}(V) \mid \underline{u} \le u(x) \le \overline{u}\}$, where

$$\mathcal{J}(u) = \frac{1}{2}\int_V |\nabla u|^2 \mathrm{d}\mu - \frac{1}{p}\int_V B(x)u^p \mathrm{d}\mu + \frac{1}{p}\int_V A(x)u^{-p}\mathrm{d}\mu.$$

Then, we can show $\widetilde{u}$ is indeed a solution to (6.1). To avoid duplication, we turn to utilize heat-flow method to derive the existence result in the sequel, in which sub- and super-solution method is also involved for heat flow.

Example 3 Thanks to (6.4), if $-A(x)/B(x) \equiv C$ for some constant $C > 0$, then (6.1) has only the constant solution $u(x) \equiv C^{\frac{1}{2p}}$ on V.

Inspired by [17, Theorem 1], we introduce the heat flow for (6.1)

$$\begin{cases} u_t - \Delta u = g(x,u), & \text{in } V \times (0,+\infty),\\ u(x,0) = u_0(x), & \text{on } V, \end{cases} \tag{6.5}$$

where $u_0(x) > 0$ on V is an arbitrary function and $g(x,u) = B(x)u^{p-1} + A(x)u^{-p-1}$ with $p > 2$. We say $u(x,t)$ is a global solution to (6.5) with initial data u_0 provided that $u(x,t) \in C^1\big([0,+\infty)\big)$ for any fixed $x \in V$, $u(x,t) \in V^{\mathbb{R}}$ for any fixed $t \in [0,+\infty)$, and $u(x,0) = u_0(x)$ on V. Similarly, the definition of local solutions only shifts the interval $[0,+\infty)$ as $[0,T]$ for $T > 0$. Some similar notations are given in Subsection 6.1.

Definition 6.1 Suppose that $\varphi_0(x,t), \psi_0(x,t) \in C^1\big([0,T]\big)$ for any fixed $x \in V$, where $T > 0$ is given.

(a) We call the bounded function $\varphi_0(x,t) > 0$ a sub-solution to (6.5) in $V \times [0,T]$ if it satisfies

$$\begin{cases} \partial_t\varphi_0 - \Delta\varphi_0 - g(x,\varphi_0) \le 0, & \text{in } V \times (0,T],\\ \varphi_0(x,0) \le u_0(x), & \text{on } V. \end{cases}$$

(b) We call the bounded function $\psi_0(x,t) > 0$ a super-solution to (6.5) in $V \times [0,T]$ if it satisfies

$$\begin{cases} \partial_t\psi_0 - \Delta\psi_0 - g(x,\psi_0) \ge 0, & \text{in } V \times (0,T],\\ \psi_0(x,0) \ge u_0(x), & \text{on } V. \end{cases}$$

Now we denote functions $\varphi_0(x,t)$ and $\psi_0(x,t)$ in $V\times[0,+\infty)$ by

$$\varphi_0(x,t):=\kappa_1\underline{u}(x),\quad \psi_0(x,t):=\kappa_2\overline{u}(x),\quad \text{for } (x,t)\in V\times[0,+\infty), \tag{6.6}$$

where $\underline{u}(x)$ and $\overline{u}(x)$ are given in (6.4), the constants $\kappa_1\in(0,1)$ and $\kappa_2\in[1,+\infty)$ are chosen such that

$$\varphi_0(x,t)\le\min_{x\in V}u_0(x),\quad \psi_0(x,t)\ge\max_{x\in V}u_0(x),\quad \text{for } (x,t)\in V\times[0,+\infty).$$

Then one can easily check that $\varphi_0(x,t)>0$ and $\psi_0(x,t)>0$ are sub- and super-solutions to (6.5) in $V\times[0,T]$ for any $T>0$ respectively. Set $\Lambda_1=\varphi_0$ and $\Lambda_2=\psi_0$. We define a map

$$\mathcal{F}=(\partial_t-\Delta+\lambda)^{-1}(g(x,\cdot)+\lambda)(\cdot):\ V\times[0,T]\to V\times[0,T],\ w\mapsto v,$$

satisfying that

$$\begin{cases}\partial_t v-\Delta v+\lambda v=g(x,w)+\lambda w, & \text{in } V\times(0,T],\\ v(x,0)=u_0(x), & \text{on } V,\end{cases} \tag{6.7}$$

where $\lambda=\lambda(A,B,u_0)>0$ is a constant large enough satisfying that $g(x,u)+\lambda u$ is increasing with respect to $u\in[\Lambda_1,\Lambda_2]$. After that, we can define two sequences $\{\varphi_k\}$ and $\{\psi_k\}$ as

$$\varphi_k=\mathcal{F}\varphi_{k-1},\quad \psi_k=\mathcal{F}\psi_{k-1},\quad \text{for } k\ge 1. \tag{6.8}$$

In any finite interval of t, we shall prove that $\{\varphi_k\}$ and $\{\psi_k\}$ are monotone increasing and decreasing respectively, and their limiting functions are identical. This establishes the short-time existence of heat-flow to (6.5). The global existence of $u(x,t)$, $(x,t)\in V\times[0,+\infty)$ follows by continuation. Finally, we need to consider the asymptotic behavior of $u(x,t)$ as $t\to+\infty$. In order to guarantee the existence of $v(x,t)$ in (6.7), we establish the existence result for general heat equation on connected finite graph.

6.1 Existence, uniqueness and maximum principle for the general heat equation

In this part, we discuss the general heat equation

$$\begin{cases}u_t-\Delta u+c(x,t)u=f(x,t), & \text{in } V\times(0,+\infty),\\ u(x,0)=u_0(x), & \text{on } V,\end{cases} \tag{6.9}$$

where $u_0(x)\in V^{\mathbb{R}}$ is an arbitrary function, $c(x,t)$ and $f(x,t)$ are $C^\infty\big([0,+\infty);V^{\mathbb{R}}\big)$ functions, that is, $c(x,\cdot),f(x,\cdot)\in C^\infty\big([0,+\infty)\big)$ for any $x\in V$, and $c(\cdot,t),f(\cdot,t)\in V^{\mathbb{R}}$ for any $t\in[0,+\infty)$. Thus, we associate $u(x,t)$ with a map

$$\mathbf{u}:[0,+\infty)\to V^{\mathbb{R}},\ \ [\mathbf{u}(t)](x):=u(x,t),\ \ x\in V,\ 0\le t<+\infty.$$

Similarly, we define

$$\begin{cases}\mathbf{f}:[0,+\infty)\to V^{\mathbb{R}}, & [\mathbf{f}(t)](x):=f(x,t),\ \ x\in V,\ 0\le t<+\infty,\\ \mathbf{c}:[0,+\infty)\to V^{\mathbb{R}}, & [\mathbf{c}(t)](x):=f(x,t),\ \ x\in V,\ 0\le t<+\infty.\end{cases}$$

Definition 6.2 For any $c(x,t),f(x,t)\in C\big([0,+\infty);V^{\mathbb{R}}\big)$, we say a function $\mathbf{u}\in C^1\big([0,+\infty);V^{\mathbb{R}}\big)$ is a solution to (6.9) provided that

(a) $[\mathbf{u}(0)](x) = u_0(x)$ for any $x \in V$;
(b) $\partial_t[\mathbf{u}(t)](x) - \Delta[\mathbf{u}(t)](x) + [\mathbf{c}(t)](x)[\mathbf{u}(t)](x) = [\mathbf{f}(t)](x)$ for any $(x, t) \in V \times (0, +\infty)$.

Theorem 6.3 *Let $G = (V, E)$ be a connected finite graph. Suppose that $c(x, t), f(x, t) \in C^\infty\big([0, +\infty); V^{\mathbb{R}}\big)$, then (6.9) admits a unique solution $\mathbf{u} \in C^\infty\big([0, +\infty); V^{\mathbb{R}}\big)$.*

Proof Supposing that $V = \{x_i\}_{1\le i\le m}$, we express any $C^\infty\big([0, +\infty); V^{\mathbb{R}}\big)$ function $u(x, t)$ as

$$u(x, t) = \sum_{i=1}^{m} d_i(t)\delta_{x_i}(x), \tag{6.10}$$

where δ_{x_i} is defined as in (2.1), the coefficient $d_i(t)$ belongs to $C^\infty\big([0, +\infty)\big)$ for $1 \le i \le m$. Thus it is equivalent to seeking for a function $u(x, t)$ of the form (6.10) satisfying that

$$d_k(0) = \sum_{i=1}^{m} d_i(0)\delta_{x_i}(x_k) = u(x_k, 0) = u_0(x_k), \quad 1 \le k \le m, \tag{6.11}$$

and

$$d_k'(t) - \frac{1}{\mu(x_k)} \sum_{i\neq k} \omega_{x_k x_i}\big(d_i(t) - d_k(t)\big) + c(x_k, t)d_k(t) = f(x_k, t), \quad t \in (0, +\infty), \quad 1 \le k \le m. \tag{6.12}$$

Applying the existence and uniqueness results of ordinary differential equations (see [24]), we obtain a unique $C^\infty\big([0, +\infty), \mathbb{R}^m\big)$ function $\mathbf{d}(t) = (d_1(t), \cdots, d_m(t))$, $t \in [0, +\infty)$ verifying (6.11) and (6.12). This completes the proof of Theorem 6.3. □

Remark 7 (a) In Theorem 6.3, if $c(x, t), f(x, t) \in C^\infty\big([0, T]; V^{\mathbb{R}}\big)$ for some $T > 0$, then (6.9) admits a unique solution $\mathbf{u} \in C^\infty\big([0, T]; V^{\mathbb{R}}\big)$.
(b) By Theorem 6.3, for any $T > 0$, we conclude that $\{\varphi_k\}$ and $\{\psi_k\}$ given by (6.7) and (6.8) are well-defined. Here we refer the readers to the arguments of **Step 1** in the proof of Theorem 2.6.

Next we present the maximum principle for the heat equation on graph.

Lemma 6.4 (Maximum principle) *Suppose that $T > 0$ and $u(x, t) \in C^1\big([0, T]; V^{\mathbb{R}}\big)$ satisfies*

$$\begin{cases} u_t - \Delta u = f(x, t), & \text{in } V \times (0, T], \\ u(x, 0) = Q(x), & \text{on } V, \end{cases}$$

where $f(x, t) \in C\big([0, T]; V^{\mathbb{R}}\big)$. Then

(a) if $f(x, t) \le 0$ in $V \times [0, T]$, we have $\max_{V\times[0,T]} u(x, t) = \max_{x\in V} Q(x)$;
(b) if $f(x, t) \ge 0$ in $V \times [0, T]$, we have $\min_{V\times[0,T]} u(x, t) = \min_{x\in V} Q(x)$.

Remark. The existence of $u(x, t) \in C^1\big([0, T]; V^{\mathbb{R}}\big)$ is guaranteed by the same arguments of Theorem 6.3.

Proof (a) We first assume that $f(x, t) < 0$ in $V \times [0, T]$ and prove this conclusion. Suppose that

$$\max_{V\times[0,T]} u(x, t) = u(x_0, t_0), \text{ for some } x_0 \in V \text{ and } 0 < t_0 \le T.$$

Then $-\Delta u(x_0, t_0) \geq 0$. If $0 < t_0 < T$, then $u_t(x_0, t_0) = 0$; if $t_0 = T$, then $u_t(x_0, t_0) \geq 0$. Thus, we have

$$0 \leq u_t(x_0, t_0) - \Delta u(x_0, t_0) = f(x_0, t_0) < 0.$$

Contradiction arises. Therefore, $t_0 = 0$ and the conclusion holds. For the general case $f(x, t) \leq 0$ in $V \times [0, T]$, we denote by $u^\varepsilon(x, t) = u(x, t) - \varepsilon t$ for $\varepsilon > 0$. Direct computation shows that

$$\begin{cases} u_t^\varepsilon - \Delta u^\varepsilon = f(x, t) - \varepsilon < 0, & \text{in } V \times (0, T], \\ u^\varepsilon(x, 0) = Q(x), & \text{on } V. \end{cases}$$

As a consequence, we deduce that

$$\max_{V \times [0,T]} u^\varepsilon(x, t) = \max_{x \in V} Q(x),$$

which implies that $\max_{V \times [0,T]} u(x, t) = \max_{x \in V} Q(x)$. In fact, one can check that

$$\begin{aligned} \max_{x \in V} Q(x) &= \max_{V \times [0,T]} u^\varepsilon(x, t) \leq \max_{V \times [0,T]} u(x, t) = \max_{V \times [0,T]} (u^\varepsilon(x, t) + \varepsilon t) \\ &\leq \max_{V \times [0,T]} u^\varepsilon(x, t) + \varepsilon T = \max_{x \in V} Q(x) + \varepsilon T \to \max_{x \in V} Q(x), \end{aligned}$$

as $\varepsilon \to 0$. This finishes the proof of (a).

(b) One can easily get it by applying (a) to $-u$.

□

Lemma 6.5 *(Maximum principle) Suppose that $T > 0$ and $u(x, t) \in C^1\big([0, T]; V^{\mathbb{R}}\big)$ satisfies*

$$\begin{cases} u_t - \Delta u + c(x, t)u = f(x, t), & \text{in } V \times (0, T], \\ u(x, 0) = Q(x), & \text{on } V, \end{cases}$$

where $f(x, t), c(x, t) \in C\big([0, T]; V^{\mathbb{R}}\big)$ and $c(x, t) \geq 0$ in $V \times [0, T]$. Then

(a) if $f(x, t) \leq 0$ in $V \times [0, T]$, we have $\max_{V \times [0,T]} u(x, t) \leq \max_{x \in V} Q^+(x)$;
(b) if $f(x, t) \geq 0$ in $V \times [0, T]$, we have $\min_{V \times [0,T]} u(x, t) \geq -\max_{x \in V} Q^-(x)$;
(c) if $f(x, t) = 0$ in $V \times [0, T]$, we have $\max_{V \times [0,T]} |u(x, t)| = \max_{x \in V} |Q(x)|$.

Proof One can prove it similarly to Lemma 6.4. □

6.2 Heat-flow method for *EL* equation

We give the proof of Theorem 2.6.

Proof of Theorem 2.6 For any $T > 0$, we see that $\varphi_0(x, t)$ and $\psi_0(x, t)$, defined in (6.6), are sub- and super-solutions to (6.5), respectively, in $V \times [0, T]$. Let $\{\varphi_k\}$ and $\{\psi_k\}$ be the sequences defined by (6.7) and (6.8). Choose $\lambda = \lambda(A, B, u_0) > 0$ large enough such that

$$\frac{\partial g(x, u)}{\partial u} + \lambda > 0 \text{ with respect to } u \in [\Lambda_1, \Lambda_2]. \tag{6.13}$$

For brevity, we split the proof into three steps.

Step 1. We establish the short-time existence result for heat flow (6.5). To this end, we first check that the sequences $\{\varphi_k\}$ and $\{\psi_k\}$ are monotone increasing and decreasing respectively. Therefore, there exists a unique u^* such that $u^* = \mathcal{F}u^*$ and

$$\lim_{k\to+\infty} \varphi_k = u^* = \lim_{k\to+\infty} \psi_k. \tag{6.14}$$

In other words, u^* is a positive solution to

$$\begin{cases} \partial_t u^* - \Delta u^* = g(x, u^*), & \text{in } V \times (0, T], \\ u^*(x, 0) = u_0(x), & \text{on } V. \end{cases} \tag{6.15}$$

Proof of Step 1 We claim that

$$\begin{cases} \varphi_k, \psi_k \in C^\infty\big([0, T]; V^{\mathbb{R}}\big) \text{ are well-defined for any } k \geq 1, \\ \varphi_0 \leq \varphi_1 \leq \cdots \leq \varphi_k \leq \cdots \leq \psi_k \leq \cdots \leq \psi_1 \leq \psi_0 \text{ in } V \times [0, T]. \end{cases} \tag{6.16}$$

In fact, by (6.7) and (6.8), $\{\varphi_k\}$ and $\{\psi_k\}$ satisfy

$$\begin{cases} \partial_t \varphi_{k+1} - \Delta\varphi_{k+1} + \lambda\varphi_{k+1} = g(x, \varphi_k) + \lambda\varphi_k, & \text{in } V \times (0, T], \\ \varphi_{k+1}(x, 0) = u_0(x), & \text{on } V, \end{cases} \tag{6.17}$$

and

$$\begin{cases} \partial_t \psi_{k+1} - \Delta\psi_{k+1} + \lambda\psi_{k+1} = g(x, \psi_k) + \lambda\psi_k, & \text{in } V \times (0, T], \\ \psi_{k+1}(x, 0) = u_0(x), & \text{on } V. \end{cases} \tag{6.18}$$

By Theorem 6.3, $\varphi_1 \in C^\infty\big([0, T]; V^{\mathbb{R}}\big)$ is well-defined. Since φ_0 is a sub-solution to (6.5), combining (6.17) with $k = 0$, we have

$$\begin{cases} \partial_t(\varphi_1 - \varphi_0) - \Delta(\varphi_1 - \varphi_0) + \lambda(\varphi_1 - \varphi_0) \geq 0, & \text{in } V \times (0, T], \\ (\varphi_1 - \varphi_0)(x, 0) \geq 0, & \text{on } V. \end{cases}$$

Applying Lemma 6.5-(b), we have $\varphi_1 \geq \varphi_0$ in $V \times [0, T]$. Again by Theorem 6.3, $\psi_1 \in C^\infty\big([0, T]; V^{\mathbb{R}}\big)$ is well-defined. Since ψ_0 is a super-solution to (6.5), combining (6.18) with $k = 0$, we deduce that $\psi_1 \leq \psi_0$ in $V \times [0, T]$ via Lemma 6.5-(a). In addition, by (6.17) and (6.18) with $k = 0$, we have

$$\begin{cases} \partial_t(\varphi_1 - \psi_1) - \Delta(\varphi_1 - \psi_1) + \lambda(\varphi_1 - \psi_1) & \\ \quad = g(x, \varphi_0) - g(x, \psi_0) + \lambda(\varphi_0 - \psi_0) \leq 0, & \text{in } V \times (0, T], \\ (\varphi_1 - \psi_1)(x, 0) = 0, & \text{on } V, \end{cases}$$

via (6.13) and $\varphi_0 \leq \psi_0$ in $V \times [0, T]$. Thus by Lemma 6.5-(a), we deduce that $\varphi_1 \leq \psi_1$ in $V \times [0, T]$.

Inductively, we assume that

$$\begin{cases} \varphi_i, \psi_i \in C^\infty\big([0, T]; V^{\mathbb{R}}\big) \text{ are well-defined for } 1 \leq i \leq k, \\ \varphi_0 \leq \varphi_1 \leq \cdots \leq \varphi_k \leq \psi_k \leq \cdots \leq \psi_1 \leq \psi_0 \text{ in } V \times [0, T]. \end{cases} \tag{6.19}$$

It remains to show

$$\begin{cases} \varphi_{k+1}, \psi_{k+1} \in C^\infty\big([0, T]; V^{\mathbb{R}}\big) \text{ are well-defined}, \\ \varphi_k \leq \varphi_{k+1} \leq \psi_{k+1} \leq \psi_k \text{ in } V \times [0, T]. \end{cases} \tag{6.20}$$

By Theorem 6.3, $\varphi_{k+1} \in C^\infty([0,T];V^{\mathbb{R}})$ is well-defined. Applying (6.17), we have

$$\begin{cases} \partial_t(\varphi_{k+1}-\varphi_k) - \Delta(\varphi_{k+1}-\varphi_k) + \lambda(\varphi_{k+1}-\varphi_k) \\ = g(x,\varphi_k) - g(x,\varphi_{k-1}) + \lambda(\varphi_k - \varphi_{k-1}) \geq 0, & \text{in } V \times (0,T], \\ (\varphi_{k+1}-\varphi_k)(x,0) = 0, & \text{on } V. \end{cases}$$

Here we have used (6.13) and the assumption (6.19). It follows from Lemma 6.5-(b) that $\varphi_{k+1} \geq \varphi_k$ in $V \times [0,T]$. Again by Theorem 6.3, $\psi_{k+1} \in C^\infty([0,T];V^{\mathbb{R}})$ is well-defined. Applying (6.13), (6.18) and (6.19), we deduce that $\psi_{k+1} \leq \psi_k$ in $V \times [0,T]$. In addition, by (6.17) and (6.18), we get

$$\begin{cases} \partial_t(\varphi_{k+1}-\psi_{k+1}) - \Delta(\varphi_{k+1}-\psi_{k+1}) + \lambda(\varphi_{k+1}-\psi_{k+1}) \\ = g(x,\varphi_k) - g(x,\psi_k) + \lambda(\varphi_k - \psi_k) \leq 0, & \text{in } V \times (0,T], \\ (\varphi_{k+1}-\psi_{k+1})(x,0) = 0, & \text{on } V. \end{cases}$$

Applying Lemma 6.5-(a), we have $\varphi_{k+1} \leq \psi_{k+1}$ in $V \times [0,T]$. Thus, (6.20) is obtained and the claim (6.16) is proved.

By claim (6.16), we are able to define

$$u^*(x,t) = \lim_{k\to+\infty} \varphi_k(x,t), \quad v^*(x,t) = \lim_{k\to+\infty} \psi_k(x,t), \quad \forall\, (x,t) \in V \times [0,T],$$

then $u^*(x,t) \leq v^*(x,t)$ and $u^*(x,t), v^*(x,t) \in [\Lambda_1, \Lambda_2]$ in $V \times [0,T]$. Letting $k \to +\infty$ in (6.17) and (6.18), we conclude that $u^*, v^* \in C^\infty([0,T];V^{\mathbb{R}})$ satisfy that

$$\begin{cases} \partial_t u^* - \Delta u^* = g(x,u^*), & \text{in } V \times (0,T], \\ u^*(x,0) = u_0(x), & \text{on } V, \end{cases} \tag{6.21}$$

and

$$\begin{cases} \partial_t v^* - \Delta v^* = g(x,v^*), & \text{in } V \times (0,T], \\ v^*(x,0) = u_0(x), & \text{on } V. \end{cases} \tag{6.22}$$

In fact, by (6.16), (6.17) and (6.18), we see that $\{\varphi_k(x,\cdot)\}$ and $\{\psi_k(x,\cdot)\}$, seen as functions of t for any fixed $x \in V$, are uniformly bounded and equicontinuous in $[0,T]$. Applying the Arzela–Ascoli Theorem, we conclude that $u^*, v^* \in C([0,T];V^{\mathbb{R}})$, and up to a subsequence, still denoted by $\{\varphi_k\}$ and $\{\psi_k\}$, we have

$$\begin{cases} \varphi_k(x,\cdot) \rightrightarrows u^*(x,\cdot) \text{ uniformly in } [0,T], \text{ as } k \to +\infty; \\ \psi_k(x,\cdot) \rightrightarrows v^*(x,\cdot) \text{ uniformly in } [0,T], \text{ as } k \to +\infty. \end{cases}$$

Differentiating (6.17) and (6.18) with respect to t, and combining (6.16), we see that $\{\partial_t\varphi_k\}$ and $\{\partial_t\psi_k\}$ are uniformly bounded and equicontinuous in $[0,T]$ as well. Again by the Arzela-Ascoli Theorem, we conclude that there exist functions $Du^*, Dv^* \in C([0,T];V^{\mathbb{R}})$, and up to a subsequence, still denoted by $\{\varphi_k\}$ and $\{\psi_k\}$, such that for any $x \in V$,

$$\begin{cases} \partial_t\varphi_k(x,\cdot) \rightrightarrows Du^*(x,\cdot) \text{ uniformly in } [0,T], \text{ as } k \to +\infty; \\ \partial_t\psi_k(x,\cdot) \rightrightarrows Dv^*(x,\cdot) \text{ uniformly in } [0,T], \text{ as } k \to +\infty. \end{cases}$$

Thus we can check that $\partial_t u^* = Du^*, \partial_t v^* = Dv^*$, and then $u^*, v^* \in C^1([0,T];V^{\mathbb{R}})$. Inductively, we get $u^*, v^* \in C^\infty([0,T];V^{\mathbb{R}})$.

Next, we claim that $u^*(x,t) \equiv v^*(x,t)$ in $V \times [0,T]$. In fact, since $u^*(x,t) \le v^*(x,t)$ in $V \times [0,T]$,

$$\begin{aligned} g(x,u^*) - g(x,v^*) &= \big((p-1)B(x)\theta_1(x,t)^{p-2} - (p+1)A(x)\theta_2(x,t)^{-p-2}\big) \\ &\quad (u^*(x,t) - v^*(x,t)) \\ &=: -\delta(x,t)(u^*(x,t) - v^*(x,t)), \end{aligned}$$

where $u^*(x,t) \le \theta_i(x,t) \le v^*(x,t)$ for $i = 1, 2$, and $\delta(x,t) \ge 0$ in $V \times [0,T]$. Subtracting (6.22) from (6.21), we get

$$\begin{cases} \partial_t(u^* - v^*) - \Delta(u^* - v^*) + \delta(x,t)(u^* - v^*) = 0, & \text{in } V \times (0,T], \\ (u^* - v^*)(x,0) = 0, & \text{on } V. \end{cases} \tag{6.23}$$

By Lemma 6.5-(c), we conclude that $u^*(x,t) \equiv v^*(x,t)$ in $V \times [0,T]$, that is, (6.14) is proved. Thus, $u^*(x,t)$ is a positive solution to (6.15) satisfying $u^*(x,t) \in [\Lambda_1, \Lambda_2]$ in $V \times [0,T]$.

The uniqueness of positive solution to (6.15) can be obtained by the same arguments of (6.23). In fact, if $\widehat{u} \in C^1\big([0,T]; V^{\mathbb{R}}\big)$ is another positive solution to (6.15), then

$$\begin{cases} \partial_t(u^* - \widehat{u}) - \Delta(u^* - \widehat{u}) = g(x,u^*) - g(x,\widehat{u}), & \text{in } V \times (0,T], \\ (u^* - \widehat{u})(x,0) = 0, & \text{on } V, \end{cases} \tag{6.24}$$

and

$$\begin{aligned} g(x,u^*) - g(x,\widehat{u}) &= \big((p-1)B(x)\theta_1(x,t)^{p-2} - (p+1)A(x)\theta_2(x,t)^{-p-2}\big) \\ &\quad (u^*(x,t) - \widehat{u}(x,t)) \\ &=: -\delta(x,t)(u^*(x,t) - \widehat{u}(x,t)), \end{aligned}$$

where $\theta_i(x,t), i = 1, 2$ are between u^* and $\widehat{u}$, and $\delta(x,t) \ge 0$ in $V \times [0,T]$. Then by Lemma 6.5-(c), we conclude that $u^*(x,t) \equiv \widehat{u}(x,t)$ in $V \times [0,T]$, as required. This finishes the proof of *Step 1*. □

Step 2. We establish the global existence result of the heat-flow (6.5). By *Step 1*, for any $T > 0$, (6.5) has a unique positive solution $u^*(x,t)$ with $u^*(x,t) \in [\Lambda_1, \Lambda_2]$ in $V \times [0,T]$. Thus, we immediately obtain the global existence result. In fact, for any $T_0 > 0$, we get the unique positive solution $u^*(x,t)$ in $V \times [0,T_0]$. We denote by $\widehat{u}(x,t)$ the unique positive in $V \times [0, T_0 + 1]$. Naturally, by the uniqueness, $u^*(x,t) \equiv \widehat{u}(x,t)$ in $V \times [0,T_0]$. Thus, the global existence follows by continuation in such a way. For the uniqueness of global existence, we just consider it in any finite interval and apply the same arguments of (6.23) or (6.24). In particular, we conclude that the global solution $u(x,t) \in C^\infty\big([0,+\infty); V^{\mathbb{R}}\big)$ satisfying

$$u^*(x,t) \in [\Lambda_1, \Lambda_2], \quad \text{for any } (x,t) \in V \times [0,+\infty). \tag{6.25}$$

Step 3. Finally, we analyze the asymptotic behavior of the global solution $u^*(x,t)$ as $t \to +\infty$. Combining (6.4), (6.6) and (6.25), we see that there exists some constant $\widetilde{C} = \widetilde{C}(A, B, u_0) > 0$ such that

$$u^*(x,t) \ge \widetilde{C}, \quad \forall\, (x,t) \in V \times [0,+\infty). \tag{6.26}$$

Indeed, we can choose $\widetilde{C} = \Lambda_1$. We can verify (6.26) in another way. For any fixed $T > 0$, set

$$u^*(x_0,t_0) := \min_{V \times [0,T]} u^*(x,t).$$

If $t_0 = 0$, then $u^*(x,t) \geq \min_{x\in V} u_0(x)$ in $V \times [0,T]$. Otherwise, $0 < t_0 \leq T$, and we have

$$0 \geq (\partial_t - \Delta)u^*(x_0,t_0) = B(x_0)(u^*(x_0,t_0))^{p-1} + A(x_0)(u^*(x_0,t_0))^{-p-1},$$

which implies that

$$u^*(x_0,t_0) \geq \left(\frac{A(x_0)}{-B(x_0)}\right)^{\frac{1}{2p}} \geq \left(\min_{x\in V}\frac{A(x)}{-B(x)}\right)^{\frac{1}{2p}} = \underline{u}.$$

Hence we conclude that

$$u^*(x,t) \geq \min\left\{\min_{x\in V} u_0(x), \underline{u}\right\} \geq \Lambda_1, \quad \forall\,(x,t) \in V \times [0,T]. \tag{6.27}$$

Since $T > 0$ is arbitrary, this gives (6.26).

For any $t > 0$, multiplying (6.5) by $\partial_t u^*$ and then integrating in $V \times [0,t]$, we have

$$\begin{aligned}\int_0^t\int_V |\partial_t u^*(x,s)|^2\mathrm{d}\mu\mathrm{d}s &+ \int_0^t\int_V(-\Delta u^*(x,s))\partial_t u^*(x,s)\mathrm{d}\mu\mathrm{d}s\\ &= \int_0^t\int_V B(x)(u^*(x,s))^{p-1}\partial_t u^*(x,s)\mathrm{d}\mu\mathrm{d}s\\ &+ \int_0^t\int_V A(x)(u^*(x,s))^{-p-1}\partial_t u^*(x,s)\mathrm{d}\mu\mathrm{d}s.\end{aligned} \tag{6.28}$$

Now we shall compute (6.28) term by term. Using integration by parts we get that

$$\begin{aligned}\int_V(-\Delta u^*(x,s))\partial_t u^*(x,s)\mathrm{d}\mu &= \int_V \Gamma(u^*(x,s),\partial_t u^*(x,s))\mathrm{d}\mu\\ &= \sum_{x\in V}\mu(x)\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u^*(y,s)-u^*(x,s))(\partial_t u^*(y,s)-\partial_t u^*(x,s))\\ &= \frac{1}{2}\sum_{x\in V}\mu(x)\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}\partial_t(u^*(y,s)-u^*(x,s))^2\\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V\Gamma(u^*)(x,s)\mathrm{d}\mu = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_V|\nabla u^*(x,s)|^2\mathrm{d}\mu.\end{aligned}$$

Hence

$$\int_0^t\int_V(-\Delta u^*(x,s))\partial_t u^*(x,s)\mathrm{d}\mu\mathrm{d}s = \frac{1}{2}\int_V|\nabla u^*(x,t)|^2\mathrm{d}\mu - \frac{1}{2}\int_V|\nabla u_0(x)|^2\mathrm{d}\mu. \tag{6.29}$$

Notice that

$$\begin{aligned}\int_V B(x)(u^*(x,s))^{p-1}\partial_t u^*(x,s)\mathrm{d}\mu &= \sum_{x\in V}\mu(x)B(x)(u^*(x,s))^{p-1}\partial_t u^*(x,s)\\ &= \frac{1}{p}\sum_{x\in V}\mu(x)B(x)\partial_t(u^*(x,s))^p = \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_V B(x)(u^*(x,s))^p\mathrm{d}\mu,\end{aligned}$$

which implies that

$$\int_0^t\int_V B(x)(u^*(x,s))^{p-1}\partial_t u^*(x,s)\mathrm{d}\mu\mathrm{d}s = \frac{1}{p}\int_V B(x)(u^*(x,t))^p\mathrm{d}\mu$$

$$-\frac{1}{p}\int_V B(x)(u_0(x))^p \mathrm{d}\mu. \tag{6.30}$$

Similarly, we can check

$$\int_0^t \int_V A(x)(u^*(x,s))^{-p-1}\partial_t u^*(x,s)\mathrm{d}\mu \mathrm{d}s = -\frac{1}{p}\int_V A(x)(u^*(x,t))^{-p}\mathrm{d}\mu$$
$$+\frac{1}{p}\int_V A(x)(u_0(x))^{-p}\mathrm{d}\mu. \tag{6.31}$$

Substituting (6.29), (6.30) and (6.31) into (6.28), we get

$$\int_0^t \int_V |\partial_t u^*(x,s)|^2 \mathrm{d}\mu \mathrm{d}s + \frac{1}{2}\int_V |\nabla u^*(x,t)|^2 \mathrm{d}\mu - \frac{1}{p}\int_V B(x)(u^*(x,t))^p \mathrm{d}\mu$$
$$+ \frac{1}{p}\int_V A(x)(u^*(x,t))^{-p}\mathrm{d}\mu$$
$$= \frac{1}{2}\int_V |\nabla u_0(x)|^2 \mathrm{d}\mu - \frac{1}{p}\int_V B(x)(u_0(x))^p \mathrm{d}\mu + \frac{1}{p}\int_V A(x)(u_0(x))^{-p}\mathrm{d}\mu.$$

Thus there exists some constant $C = C(A, B, u_0) > 0$ such that for any $t > 0$,

$$\int_V (u^*(x,t))^p \mathrm{d}\mu \le C, \quad \int_0^t \int_V |\partial_t u^*(x,s)|^2 \mathrm{d}\mu \mathrm{d}s \le C. \tag{6.32}$$

Then by Lemma 3.1-(b), there exists a function $u_\infty \in L^\infty(V)$ and a sequence $\{t_k\}$ with $t_k \to +\infty$ as $k \to +\infty$ such that

$$\int_V |\partial_t u^*(x,t_k)|^2 \mathrm{d}\mu \to 0, \quad u^*(x,t_k) \to u_\infty(x) \text{ in } L^\infty(V), \quad \text{as } k \to +\infty.$$

Letting $k \to +\infty$ in (6.5) with $t = t_k$, and combining (6.26), we see u_∞ is a point-wise positive solution to the *EL* Eq. (6.1), that is, (2.9) holds. This completes the proof of Theorem 2.6. □

6.3 Topological degree

In this part, we calculate the topological degree for Eq. (6.1).

Lemma 6.6 *Suppose that $A(x) > 0$, $B(x) < 0$ on V, and $\{u_n\}$ is a sequence of positive solutions to (6.1), that is,*

$$-\Delta u_n(x) = B_n(x)u_n(x)^{p-1} + A_n(x)u_n(x)^{-p-1}, \quad \forall\, x \in V,$$

where $\{A_n\}$ and $\{B_n\}$ satisfy that

$$\lim_{n\to+\infty} A_n(x) = A(x), \quad \lim_{n\to+\infty} B_n(x) = B(x), \quad \forall\, x \in V.$$

Then up to a subsequence (still denoted by $\{u_n\}$), we have $\{u_n\}$ is bounded in $L^\infty(V)$.

Proof By Remark 6, the conclusion is trivial. □

Finally, we give the proof of Theorem 2.7.

Proof of Theorem 2.7 Suppose that $\max_{x\in V} A(x) = A_0$ and $\min_{x\in V} B(x) = B_0$. Let $\{u_t\}_t$, $t \in [0, 1]$ satisfy

$$-\Delta u_t = \big((1-t)B + tB_0\big)u_t^{p-1} + \big((1-t)A + tA_0\big)u_t^{-p-1} \text{ on } V. \tag{6.33}$$

By Lemma 6.6 or Remark 6, $\{u_t\}$ is uniformly bounded in $L^\infty(V)$. We omit the details and refer the readers to the same arguments of Theorem 2.3-(c). So the topological degree $\mathbf{d}_{0,A,B}$ is well-defined. By the homotopy invariance, we have $\mathbf{d}_{0,A,B} = \mathbf{d}_{0,A_0,B_0}$. Noticing that by Remark 6 or Example 3, $u_1(x) \equiv (-A_0/B_0)^{\frac{1}{2p}}$ is the unique positive solution to (6.33) with $t = 1$. Therefore, we obtain

$$\mathbf{d}_{0,A,B} = \mathbf{d}_{0,A_0,B_0} = \text{sgn}\det\big(D\mathcal{A}_{0,A_0,B_0}(u_1)\big) = 1.$$

In fact, direct computation shows that

$$D\mathcal{A}_{0,A_0,B_0}(u_1) =$$
$$\begin{pmatrix} 1+2pA_0^{\frac{p-2}{2p}}(-B_0)^{\frac{p+2}{2p}} & -\frac{\omega_{x_1x_2}}{\mu(x_1)} & -\frac{\omega_{x_1x_3}}{\mu(x_1)} & \cdots & -\frac{\omega_{x_1x_m}}{\mu(x_1)} \\ -\frac{\omega_{x_2x_1}}{\mu(x_2)} & 1+2pA_0^{\frac{p-2}{2p}}(-B_0)^{\frac{p+2}{2p}} & -\frac{\omega_{x_2x_3}}{\mu(x_2)} & \cdots & -\frac{\omega_{x_2x_m}}{\mu(x_2)} \\ -\frac{\omega_{x_3x_1}}{\mu(x_3)} & -\frac{\omega_{x_3x_2}}{\mu(x_3)} & 1+2pA_0^{\frac{p-2}{2p}}(-B_0)^{\frac{p+2}{2p}} & \cdots & -\frac{\omega_{x_3x_m}}{\mu(x_3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\omega_{x_mx_1}}{\mu(x_m)} & -\frac{\omega_{x_mx_2}}{\mu(x_m)} & -\frac{\omega_{x_mx_3}}{\mu(x_m)} & \cdots & 1+2pA_0^{\frac{p-2}{2p}}(-B_0)^{\frac{p+2}{2p}} \end{pmatrix},$$

which is a strictly diagonally dominant and symmetric matrix whose principal diagonal elements are positive. Hence it is positive definite. □

6.4 Detailed conclusions derived by heat flow

We consider the heat flow (6.5) with $B(x) \equiv -1$ and $A(x) \equiv 1$ on V, that is,

$$\begin{cases} u_t - \Delta u = g(x,u) := -u^{p-1} + u^{-p-1}, & \text{in } V \times (0,+\infty), \\ u(x,0) = u_0(x), & \text{on } V, \end{cases} \tag{6.34}$$

where $p > 2$ and $u_0(x) > 0$ is a given function on V.

Corollary 6.7 *(a) If $0 < u_0(x) \le 1$ on V, then the positive solution $u(x,t)$ to Eq. (6.34) satisfies that*

$$u(x,t) \to 1 \textit{ uniformly on } V, \textit{ as } t \to +\infty.$$

(b) If $0 < u_0(x) \le L$ on V for some $L > 1$, then the positive solution $u(x,t)$ to Eq. (6.34) satisfies that

$$u(x,t) \to 1 \textit{ uniformly on } V, \textit{ as } t \to +\infty.$$

Proof (a) By Theorem 2.6, the global solution $u(x,t) \in C^\infty\big([0,\infty); V^{\mathbb{R}}\big)$ is obtained for (6.34). For any $T > 0$, let $(x',t') \in V \times [0,T]$ be such that $u(x',t') = \max_{V\times[0,T]} u(x,t)$. Thus either

(i) $t' = 0$, then $u(x,t) \le \max_{x\in V} u_0(x) \le 1$ in $V \times [0,T]$; or
(ii) $t' > 0$, then $\partial_t u(x',t') \ge 0$, $-\Delta u(x',t') \ge 0$, and hence $-\big(u(x',t')\big)^{p-1} + \big(u(x',t')\big)^{-p-1} \ge 0$, which implies that $u(x',t') \le 1$.

Combining the above arguments and (6.27), we get $\min_{x\in V} u_0(x) \le u(x,t) \le 1$ in $V \times [0,T]$.
Next, we check that $u(x,t) \to 1$ uniformly on V as $t \to +\infty$. We define

$$\begin{cases} u_{\min}(t) = \min\limits_{x\in V} u(x,t) = \min\left\{u(x_i,t) \mid 1 \le i \le m\right\} = u(x_t,t), & \text{for } t \in (0,+\infty), \\ u_{\max}(t) = \max\limits_{x\in V} u(x,t) = \max\left\{u(x_i,t) \mid 1 \le i \le m\right\} = u(x_t',t), & \text{for } t \in (0,+\infty), \end{cases}$$

where $x_t, x_t' \in V$. Then the upper derivatives of $u_{\min}$ and $u_{\max}$ are given by

$$D^+ u_{\min/\max}(t) = \limsup_{h\to 0^+} \frac{u_{\min/\max}(t+h) - u_{\min/\max}(t)}{h}, \quad \forall\, t > 0.$$

One can check that $u_{\min}$ is locally Lipschitz continuous in $(0,+\infty)$ since $u(x,t) \in C^\infty\big([0,+\infty); V^{\mathbb{R}}\big)$. Without loss of generality, we may assume that $u_{\min}$ is differentiable in $(0,+\infty)$. If there exists some $\varepsilon_0 > 0$ such that $u_{\min}(t) \le 1 - \varepsilon_0$ for any $t > 0$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{\min} \ge -u_{\min}^{p-1} + u_{\min}^{-p-1} \ge \big(-(1-\varepsilon_0)^{p-2} + (1-\varepsilon_0)^{-p-2}\big) u_{\min} \ge C(\varepsilon_0) u_{\min} > 0,$$

which yields that $u_{\min}(t) \ge \exp\big(C(\varepsilon_0)t\big) u_{\min}(0) \to +\infty$ as $t \to +\infty$. Contradiction arises. Thus for any $\varepsilon > 0$ and any $x \in V$, there exists some $T = T(\varepsilon) > 0$ such that $u(x,t) \ge 1 - \varepsilon$ whenever $t > T$. We conclude that $u(x,t) \to 1$ uniformly on V as $t \to +\infty$.

(b) By (6.27), the global solution $u(x,t) \in C^\infty\big([0,+\infty); V^{\mathbb{R}}\big)$ to (6.34) satisfies

$$u(x,t) \ge \min_{x\in V} u_0(x) > 0, \quad \text{for any } (x,t) \in V \times [0,+\infty).$$

For any $T > 0$, let $(x',t') \in V \times [0,T]$ be such that $u(x',t') = \max_{V\times[0,T]} u(x,t)$. Thus, either

(i) $t' = 0$, then $u(x',t') = \max_{x\in V} u_0(x)$; or

(ii) $t' > 0$, then $\partial_t u(x',t') \ge 0$, $-\Delta u(x',t') \ge 0$ and hence $-\big(u(x',t')\big)^{p-1} + \big(u(x',t')\big)^{-p-1} \ge 0$, which implies that $u(x',t') \le 1$.

Therefore, if $\max_{x\in V} u_0(x) \le 1$, we get $u(x,t) \le 1$ in $V \times [0,T]$, and the conclusion (b) holds directly by (a). So, without loss of generality, we assume that $\max_{x\in V} u_0(x) \in (1,L]$. In addition, if $u_{\max}(t_0) \le 1$ for some $t_0 > 0$, then we can derive that $u_{\max}(t) \le 1$ for any $t \ge t_0$ by repeating almost the same argument. Then using conclusion (a) we get the desired conclusion. Therefore, without loss of generality we may assume that $1 < u_{\max}(t) \le L$ for any $t \in [0,+\infty)$.
Notice that $u(x) \equiv 1$ is indeed the positive solution to (6.1) with $B(x) \equiv -1$ and $A(x) \equiv 1$ on V. Then by Theorem 2.6 and the same arguments of (a), it remains to show that for any $x \in V$,

$$u(x,t) \to 1, \quad \textit{as } t \to +\infty. \tag{6.35}$$

In fact, if there exists some $\varepsilon_0 > 0$ such that $u_{\min}(t) \le 1 - \varepsilon_0$ for all $t > 0$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{\min} \ge C(\varepsilon_0) u_{\min} > 0,$$

which implies that $u_{\min}(t) \ge \exp\big(C(\varepsilon_0)t\big) u_{\min}(0) \to +\infty$ as $t \to +\infty$. Contradiction arises. Hence for any $\varepsilon > 0$ and any $x \in V$, there exists some $T = T(\varepsilon) > 0$ such that $u(x,t) \geq 1-\varepsilon$ whenever $t > T$. Similarly, if there exists some $\varepsilon_0 > 0$ such that $u_{\max}(t) \geq 1+\varepsilon_0$ for all $t > 0$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t} u_{\max} \leq C(\varepsilon_0) u_{\max} < 0,$$

which implies that $1 < u_{\max}(t) \leq \exp\big(C(\varepsilon_0)t\big)u_{\max}(0) \to 0$ as $t \to +\infty$, a contradiction. Hence for any $\varepsilon > 0$ and any $x \in V$, there exists some $T' = T'(\varepsilon) > 0$ such that $u(x,t) \leq 1+\varepsilon$ whenever $t > T'$. Combining the above two parts, we obtain (6.35). This finishes the proof of Corollary 6.7.

□

Corollary 6.8 *Suppose that $-A(x)/B(x) \equiv C$ on V for some constant $C > 0$. For any positive initial data $u_0(x)$, let $u(x,t)$ be the unique positive solution to* (6.5). *Then $u(x,t) \to C^{\frac{1}{2p}}$ uniformly on V as $t \to +\infty$.*

Proof The proof of Corollary 6.8 is similar to Corollary 6.7. □

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